On measurement point-independent identification of maxwell model of viscoelastic materials

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Summary. The problem of a weighted least-squares approximation of viscoelastic material by generalized Maxwell model is discussed when only the noise-corrupted time-measurements of the relaxation modulus are accessible for identification. To build a Maxwell model, which does not depend on sampling instants is a basic concern. It is shown that even when the true relaxation modulus description is completely unknown, the approximate optimal Maxwell model parameters can be derived from the measurement data sampled randomly according to appropriate randomization. The determined approximate model is a strongly consistent estimate of the requested model. An identification algorithm leading to the best model will be presented in the forthcoming paper, in which the convergence analysis will be also conducted. A motivating example is given.

Key words: viscoelasticity, relaxation modulus, Maxwell model, model identification

INTRODUCTION

Viscoelastic materials present a behaviour that implies dissipation and storage of mechanical energy. Viscoelastic models are used before all to modelling of different polymeric liquids and solids [3, 6, 14], concrete [1], soils [13], rubber [30], glass [5, 23], foods [2, 20, 22, 24], biological materials [8, 9, 16], soda-lime-silica glass [5]. Next, the application of Maxwell model to compute other material functions such as the creep compliance, time-variable bulk and shear modulus or time-variable Poisson’s ratio [27] or interconversion between linear viscoelastic material functions [19]. And finally, the development of computational tools for Maxwell model determining [21, 30]. This paper belongs to the latter group.

We often determine the parameters in a model by obtaining the „best-possible” fit to experimental data. The coefficients can be highly dependent on our way of measuring „best” [17]. Common choice of the model quality measure is the mean square approximation error, leading to a least-squares identification problem. When the identification index is fixed, the coefficients can be also highly dependent on the measurement data. To make the idea a little clear we give an example of the four-parameter Maxwell model determination of an confined cylindrical specimen of the beet sugar root.

To build a Maxwell model, which does not depend on sampling instants is a basic concern. We consider the problem of measurement point-independent approximation of a linear relaxation modulus of viscoelastic material within
the class of discrete generalized Maxwell models when
the integral weighted square error is to be minimized
and the true material description is completely unknown.
We show how the problem can be solved by introducing
an appropriate randomization on the set of sampling
instants at which the relaxation modulus of the material is
measured. It is assumed that only the relaxation modulus
measurements are accessible for identi-
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the
vector of model (2) parameters is de-
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eries with a dashpot. This model presents a relaxation of ex-
dfi
used to describe the relaxation modulus
for any given value of the time t
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consider a linear viscoelastic material subjected
to small deformations for which the uniaxial, nonagin-
g and isotropic stress-strain equation can be represented
by a Boltzmann superposition integral [6]:
\[ \sigma(t) = \int_0^T G(t - \lambda) \dot{e}(\lambda) d\lambda, \] (1)
where: \( \sigma(t) \) and \( \dot{e}(t) \) denotes the stress and strain, re-
respectively, and \( G(t) \) is the linear time-dependent relaxation
modulus. The modulus \( G(t) \) is the stress, which is
induced in the viscoelastic material described by equation
(1) when the unit step strain \( \dot{e}(t) \) is imposed.
By assumption, the exact mathematical description of the relaxation modulus \( G(t) \) is completely unknown, but the value of \( G(t) \) can be measured with a certain accuracy
for any given value of the time \( t \in T \), where \( T = [0, T] \) and
0<T<\infty or \( T = R_+ \); here \( R_+ = [0, \infty) \).
MAXWELL MODEL
The generalized discrete Maxwell model, which is used
to describe the relaxation modulus \( G(t) \), consists of a spring and n Maxwell units connected in parallel as illustrated in Figure 1. A Maxwell unit is a series arrangement of the Hooke and Newton’s elements: an ideal spring in series with a dashpot. This model presents a relaxation of ex-
ponential type given by a finite Dirichlet-Prony series [29]:
\[ G_M(t, \mathbf{g}) = \sum_{j=1}^{N} E_j e^{-\eta_j t} + E_\infty, \] (2)
where: \( E_j \), \( \eta_j \) and \( E_\infty \) represent the elastic modulus
(relaxation strengths), relaxation frequencies and equi-
librium modulus (long-term modulus), respectively. The vector of model (2) parameters is defined as:
\[ \mathbf{g} = [E_1 \ldots E_n \ \eta_1 \ldots \eta_n \ E_\infty]^T. \] (3)
The modulus \( E_j \) and the viscosity \( \eta_j \) associated with the \( j-th \) Maxwell mode (see Figure 1) determine the relaxation frequency \( \nu_j = E_j / \eta_j \).

IDENTIFICATION OF THE MAXWELL MODEL

MAXWELL MATERIAL

We show how the problem can be solved by introducing
an appropriate randomization on the set of sampling
instants at which the relaxation modulus of the material is
measured. It is assumed that only the relaxation modulus
measurements are accessible for identi-
fication. -The idea of measurement point-independent identifica-
tion was at first used for noiseless zero-memory system approxima-
tion by random choice of inputs in co-author paper [12].

IDENTIFICATION OF THE MAXWELL MODEL

A classical manner of studying viscoelasticity is by
two-phase stress relaxation test, where the time-dependent shear stress is studied for step increase in strain [20, 29]. Suppose, a certain stress relaxation test performed on the specimen of the material under investigation resulted in a set of measurements of the relaxation modulus \( \tilde{G}(t_i) = G(t_i) + z(t_i) \) at the sampling instants \( t_i \geq 0 \),
i = 1, ..., N, where \( z(t) \) is measurement noise. Identification consists of selecting within the given class of models
(2), (3) a model, which ensures the best fit to the measurement results. As a measure of the model (2) accuracy the mean sum of squares is taken:
\[ Q_N(\mathbf{g}) = \frac{1}{N} \sum_{i=1}^{N} (\tilde{G}(t_i) - G_M(t_i, \mathbf{g}))^2. \] (4)

This is the least-squares criterion for Maxwell model. Therefore the least-squares Maxwell model identification consists of determining the parameter \( \mathbf{g}_N \) minimizing the index (4) on the set \( \mathbf{G} \) by solving the following optimization problem:
\[ Q_N(\mathbf{g}_N) = \min_{\mathbf{g} \in \mathbf{G}} Q_N(\mathbf{g}). \] (5)

Exponential sum models are used frequently in applied research: time series in economics, biology, medicine, heat diffusion and diffusion of chemical compounds in engineering and agriculture, physical sciences and technology, see, e.g., [7, 18]. Fitting data to exponen-
tial sums is a very old problem, which has been studied for a long time. Several articles have appeared mainly to finding optimal least-squares exponential sum approximations to sampled data. Holmström and Peters-
son [15] have reviewed known algorithms in much detail.

The results of identification, both the model parameters and the resulting relaxation modulus are (strongly) dependent on the measurement data, in particular of the sampling instants \( t_i \). This is best illustrated by an example.
EXPERIMENT AND MOTIVATING EXAMPLE

A cylindrical sample of 20 mm diameter and height was obtained from the root of sugar beet Janus variety [9]. During the two-phase stress relaxation test performed by Golacki and co-workers at the University of Life Sciences in Lublin [9], in the first initial phase the strain was imposed instantaneously, the sample was preconditioned at the 0.5 m·s\(^{-1}\) strain rate to the maximum strain. Next, during the second phase at constant strain the corresponding deformation; i.e. the specimen examined underwent deformation in steel cylinder (for details see, for example, [9]). The modelling of mechanical properties of this material in linear-viscoelastic regime is justified by the research results presented in a lot of works, for example [8]. The experiment was performed in the state of uniaxial deformation; i.e. the specimen examined underwent deformation, and the four parameter Maxwell models:

\[ G_M(t) = E_1 e^{-\nu_1 t} + E_2 e^{-\nu_2 t}, \]  

where the elastic modulus \( E \), and the relaxation frequencies \( \nu_i \), \( i=1,2 \), were determined for each \( N \). The results of the identification, i.e. the optimal model parameters and the optimal values of the empirical index \( Q_N(g) \) are given in Table 1.

### Table 1. Maxwell model (6) parameters and the values of identification index \( Q_N(g) \); equidistant-experiment

<table>
<thead>
<tr>
<th>( N )</th>
<th>( Q_N(g) )</th>
<th>( E_{x_1} ) [MPa]</th>
<th>( E_{x_2} ) [MPa]</th>
<th>( \nu_{x_1} ) [s(^{-1})]</th>
<th>( \nu_{x_2} ) [s(^{-1})]</th>
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<tr>
<td>15</td>
<td>0.0025</td>
<td>10.563</td>
<td>3.9242</td>
<td>8.6049E-4</td>
<td>1.7521</td>
</tr>
<tr>
<td>20</td>
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<td>10.5135</td>
<td>3.8923</td>
<td>7.8748E-4</td>
<td>0.6296</td>
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<tr>
<td>25</td>
<td>0.0021</td>
<td>10.5368</td>
<td>3.8923</td>
<td>8.5423E-4</td>
<td>0.7776</td>
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<tr>
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<td>10.5372</td>
<td>3.9074</td>
<td>8.0999E-4</td>
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</tr>
<tr>
<td>50</td>
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<td>3.9133</td>
<td>8.3473E-4</td>
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</tr>
<tr>
<td>75</td>
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<td>8.4544E-4</td>
<td>2.1065</td>
</tr>
<tr>
<td>100</td>
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<td>3.9946</td>
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<td>2.6723</td>
</tr>
<tr>
<td>150</td>
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<td>4.2195</td>
</tr>
<tr>
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<tr>
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<tr>
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<tr>
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</tr>
</tbody>
</table>

Fig. 2. (a) The distance \( d_N \), between the two successive Maxwell model parameters \( g_{x_1} \) and (b) the identification index \( Q_N(g) \) as a function of the number of measurements \( N \). To illustrate the convergence of the Maxwell model parameters in Figure 2(a) the distance \( d_N = ||g_{x_1} - g_{x_2}||_2 \), where \( [N] \) is a direct predecessor of \( N \) in the set \( N \), is shown as a function of \( N \); here \( || \cdot || \) denotes the Euclidean norm in the space \( R^{2m} \). The course of the model quality index as a function of \( N \) is illustrated in Fig. 2(b). The relaxation modulus computed according to the best ‘equidistant’ Maxwell models \( G_M(t,g_{x_1}) \) are plotted in Figure 3 for a few values of the number of measurements, where the measurements \( G_M(t) \) are also marked. However, the models \( G_M(t,g_{x_1}) \) does not differ significantly (see Figure 3), the model parameters differ essentially - compare Figure 2(a) and Table 1.

The above example illustrates that the Maxwell model parameters will be highly dependent on the measurement data, if the sampling instants \( t_i \) are inappropriately chosen. This is a crucial point of the problem. Loosely speaking, the problem is, whether the identification procedure will
yield a Maxwell model parameters which are asymptotically (when the number of measurements tends to infinity) independent on the particular sampling instants. The issue involves aspects on whether the data set (i.e. the experimental conditions) is informative enough to guarantee this convergence result. We show, that this problem can be satisfactorily solved by introducing a simple randomization on the sampling times set.

**METHODS AND RESULTS**

**OPTIMAL APPROXIMATION OF THE MAXWELL MODEL**

As a measure of the model (2), (3) accuracy the global approximation error of the form:

$$Q(g) = \int_T \left[ G(t) - G_M(t, g) \right]^2 \rho(t) dt,$$

(7)

where a chosen weighting function $\rho(t)\geq 0$ is a density on $T$, i.e., $\int \rho(t) dt = 1$, can be taken. Thus, the problem of the real relaxation modulus $G(t)$ optimal approximation within the class of Maxwell models reduces, obviously, to determining the parameter $g^*$ minimizing the index $Q(g)$ on the set of admissible parameters $G$, i.e. takes the form:

$$g^* = \operatorname{arg min}_{g \in G} Q(g),$$

(8)

where $\operatorname{arg min}_{g \in G} Q(g)$ denotes the vector $g$ that minimizes $Q(g)$ on the set $G$. Note, that the empirical index $Q_N(g)$ (4) is obtained by the replacement of the integral in $Q(g)$ with the finite mean sum of squares.

**MATHEMATICAL BACKGROUND AND ASSUMPTIONS**

Let $T_1, \ldots, T_N$ are independent random variables with a common probability density function $\rho(t)$ whose support is $T$. Let $G_i = G(T_i)$ be the corresponding relaxation modulus, $i=1, \ldots, N$, and let $G_i = G_i + Z_i = G(T_i) + Z_i$ denote

![Fig. 3. The relaxation modulus measurements $\tilde{G}(t_i)$ (points) and the approximate Maxwell models $G_M(t, g_N)$ (solid line); equi-distant-experiment](image)
their measurements obtained in a certain stress relaxation test performed on the specimen of the material under investigation. Here $Z$ are additive measurement noises.

We take the following assumptions, which seems to be quite natural in the context of relaxation modulus approximation task.

- **Assumption 1.** The relaxation modulus $G(t)$ is bounded on $\mathcal{T}$, i.e. $\sup_{t \in \mathcal{T}} G(t) < \infty$.
- **Assumption 2.** The set of admissible model parameters $\mathcal{G}$ is compact in the space $R^{2n+1}$.
- **Assumption 3.** The measurement noises $Z_i$ are bounded, i.e. $|Z_i| \leq \delta < \infty$, for $i = 1, \ldots, N$.
- **Assumption 4.** The sequence $\{Z_i\}$ is a time-independent sequence of independent identically distributed (i.i.d.) random variables with zero mean and a common finite variance $\sigma^2$: $E[Z_i] = 0$ and $E[Z_i^2] = \sigma^2 < \infty$.

Note that the assumption 1 is satisfied, in particular, if $G(0) < \infty$ and the weak energy dissipation principle is satisfied – for details see, for example [25]. Obviously, from assumption 4 if follows that $E[G(T) + Z_i - G_{\mu}(T, \mathbf{g})] = O(g) + \sigma$. Taking into account the Maxwell model equations (2), (3) set-up we see that the following propositions hold.

- **Property 1.** $G_{\mu}(t, \mathbf{g})$ is continuous and differentiable with respect to $\mathbf{g}$ for any $t \in \mathcal{T}$.
- **Property 2.** $\sup_{t \in \mathcal{T}} \|\nabla G_{\mu}(t, \mathbf{g})\| < \infty$ for any arbitrary compact subset $\mathcal{G}$ of $R^{2n+1}$.
- **Property 3.** $\sup_{t \in \mathcal{T}} \|G_{\mu}(t, \mathbf{g})\| < \infty$ for any arbitrary compact subset $\mathcal{G}$ of the space $R^{2n+1}$.

Notice that, since in view of Proposition 1 the quality indices $O(g)$ and $Q_\delta(g)$ are continuous with respect to $\mathbf{g}$, then if the set $\mathcal{G}$ is compact in the space $R^{2n+1}$, the solutions of the optimal approximation tasks (5) and (8) there exist, on the basis of the well-known Weierstrass’s theorem which asserts the existence of continuous function extrema on compact sets.

**ASYMPTOTIC PROPERTIES OF THE OPTIMAL MODEL**

Now we wish to investigate the stochastic-type asymptotic properties of the Maxwell model approximation tasks (5) and (8). When studying these issues, the following proposition is instrumental.

- **Proposition 1.** When the relaxation modulus measurements are corrupted by additive noise and the Assumptions 1-4 are satisfied, then:

$$\sup_{g \in \mathcal{G}} \left| O(g) + \sigma^2 - Q_\delta(g) \right| \rightarrow 0 \quad \text{w.p.1 as } N \rightarrow \infty,$$

where w.p.1 means “with probability one”.

The proof follows immediately from Property 2 in [12]. To verify this claim we need only note that the above Properties 1 and 2 guarantee that the assumptions A2 and A3 in [12] are satisfied. Next, the Assumption 2 is equivalent to A1, the Assumption 4 is equivalent to A5 ibidem, and due to Assumption 1 and Property 3 the assumption A4 ibidem holds.

Proposition 1 enables us to relate the Maxwell model parameter $\mathbf{g}_\delta$ solving the optimal approximation task (5) for empirical index $Q_\delta(g)$ to the parameter $\mathbf{g}^*$ minimizing the deterministic function $Q_{\mu}(g)$ in (8). Namely, from the uniform in $g \in \mathcal{G}$ convergence of the index $Q_{\delta}(g)$ in (9) we conclude immediately the following.

- **Proposition 2.** Assume that Assumptions 1-4 are in force, $T_i, \ldots, T_N$ being independently, at random selected from $\mathcal{T}$, each according to probability distributions with density $p(t)$. Then for the additive noise corrupted relaxation modulus measurements:

$$g_N \rightarrow g^* \quad \text{w.p.1 as } N \rightarrow \infty \quad (10)$$

and for all $t \in \mathcal{T}$:

$$G_M(t, g_N) \rightarrow G_M(t, g^*) \quad \text{w.p.1 as } N \rightarrow \infty.$$

Thus, under the taken assumptions the Maxwell model parameter $\mathbf{g}_\delta$ is strongly consistent estimate of the parameter $\mathbf{g}^*$. Moreover, since the Maxwell model $G_{\mu}(t, \mathbf{g})$ is Lipschitz on $\mathcal{G}$ uniformly in $t \in \mathcal{T}$ (the above is guaranteed by Property 2), then the almost sure convergence of $\mathbf{g}_\delta$ to the respective parameter $\mathbf{g}^*$ in (10) implies that:

$$\sup_{t \in \mathcal{T}} \|G_{\mu}(t, g_N) - G_M(t, g^*)\| \rightarrow 0 \quad \text{w.p.1 as } N \rightarrow \infty,$$

i.e., $G_{\mu}(t, g_N)$ is in the case considered a strongly uniformly consistent estimate of the best model $G_{\mu}(t, g^*)$.

**CONCLUSIONS**

Summarizing, when the Assumptions 1-4 are satisfied, the arbitrarily precise approximation of the optimal Maxwell model (with the parameter $\mathbf{g}^*$) can be obtained (almost everywhere) as the number of measurements $N$ grows large, despite the fact that the real description of the relaxation modulus is completely unknown. Thus, when the set $\{t_i\}$ is open to manipulation during the data collection, it is an important experiment design issue to take an appropriate sampling instants. We shall comment on how to do this in the forthcoming paper [26], where the complete identification algorithm providing the strongly consistent estimate of the optimal model is given. The stochastic-type convergence analysis is also performed in [26] and the rate of convergence is discussed for the case when the measurements are perfect or corrupted by additive noises.

**REFERENCES**


O NIEZALEŻNEJ OD PUNKTÓW POMIAROWYCH IDENTYFIKACJI MODELU MAXWELLA

MATERIAŁÓW LEPKOSPRĘŻYSTYCH

Odpowiedni algorytm identyfikacji będzie przedmiotem kolejnej pracy, w której przeprowadzona zostanie także analiza zbieżności modelu.

Słowa kluczowe: lepkosprężystość, moduł relaksacji, model Maxwella, identyfikacja modelu.