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MODELLING OF PASSIVE VIBRATION DAMPING USING PIEZOELECTRIC TRANSDUCERS – THE MATHEMATICAL MODEL

MODELOWANIE PASYWNEGO TŁUMIENIA DRGAŃ PRZY UŻYCIU PRZETWORNIKÓW PIEZOELEKTRYCZNYCH – MODEL MATEMATYCZNY*

A proposal of mathematical modelling of vibrating piezoelectric transducers using the electrical analogy is presented in this work. The developed mathematical model is used in analysis of vibrating piezoelectric plates with adjoined external passive electric elements and for designating of their characteristics. A substitute electric system of the piezoelectric transducer that is equal to a three-port system was introduced. A piezoelectric transformer created by connection of two plates was analysed. Substitute systems of both plates were introduced. All mechanical parameters of the analysed system were replaced by equivalent electrical parameters in obtained Mason's model.

Keywords: *piezoelectric transducers, piezoelectric transformer, vibration damping, passive electric circuits.*

W pracy przedstawiono propozycję modelowania matematycznego drgających przetworników piezoelektrycznych poprzez zastosowanie analogii elektrycznej. Opracowany model stosowany jest w analizie oraz wyznaczaniu charakterystyk drgających płytek piezoelektrycznych z dołączonymi, zewnętrznymi, biernymi elementami elektrycznymi. Wprowadzono układ zastępczy przetwornika piezoelektrycznego równoważny trójwrotnikowi elektrycznemu. W pracy analizowano połączenia dwóch płytek piezoelektrycznych działających jako transformator piezoelektryczny, wprowadzając układy zastępcze obu przetworników. Przy stosowaniu układów zastępczych w postaci obwodów elektrycznych wszystkie wielkości mechaniczne w otrzymanym układzie Masona zostały zastąpione równoważnymi wielkościami elektrycznymi.

Słowa kluczowe: *przetworniki piezoelektryczne, transformator piezoelektryczny, tłumienie drgań, modelowanie, pasywne obwody elektryczne.*

1. Introduction

Piezoelectric elements are increasingly used in modern technical means. The reason for the growth in popularity of this type of materials are, inter alia, high efficiency of conversion of electrical to mechanical energy or in the opposite direction and the possibility to produce a piezoelectric transducers of any shape, suitable for the application. One of the possible applications of piezoelectric transducers are systems with passive damping of mechanical vibrations with external electric circuits adjoined to the piezoelectric transducers [9–9, 18, 19].

In the Gliwice Research Centre works which aim is to develop a mathematical algorithms for analysis and determination of characteristics for both vibrating mechanical systems and mechatronic systems containing piezoelectric transducers used as vibration dampers or actuators are realized [1–14, 25, 26, 28]. Issues of synthesis of such kind of systems, so their design taking into account required characteristics, as well as computer-aided methods of synthesis and analysis are also considered [3–5, 10, 12, 16, 17].

Development of mathematical models of this type of systems with high detail representation of real systems is a very important issue due to the complexity of the phenomena occurring in them. The correct description of the system in the form of mathematical model already in the design phase is a important condition, necessary to obtain desired results, such as required characteristics of the system, as well as the maximum efficiency of its operation [7,21,22]. This work is therefore an introduction to the process of modelling of vibrating piezoelectric plates with passive electric circuits attached in order to damp vibrations. This paper proposes a description of the piezoelectric vibrating plate as a substitute electric circuit. Then the developed scheme was extended to describe two plates that interact with each other acting as a piezoelectric transformer. Using the electromechanical analogy, the dynamic flexibility of the system was determined and presented on chart. In further studies the created mathematical algorithm will be used to model and determine characteristics of systems with adjoined electric circuits. It will be also used to analyze the impact of the system's parameters on obtained characteristics.

2. The substitute scheme of the piezoelectric transducer

A mathematical model of longitudinally vibrating piezoelectric plate is considered in this work. The form of a single, free piezoelectric plate loaded by external forces F_1 and F_2 are presented in Fig. 1. Symbols u_1 and u_2 denote displacements of the transducer's surfaces and U denotes the electric voltage generated by the transducer as a result of its deformation [14].

Symbols a, b and d denote geometric dimensions of the considered system and Δd denotes the plate's thickness change.

A substitute electric system of the piezoelectric transducer that is equal to a three-port system was introduced. It is a system with three pairs of terminals corresponding to the mechanical and electri-

(*) Tekst artykułu w polskiej wersji językowej dostępny w elektronicznym wydaniu kwartalnika na stronie ***www.ein.org.pl*** cal inputs or outputs. A block diagram of this system together with the indication of the stiffness matrix is shown in Fig. 2 [24].

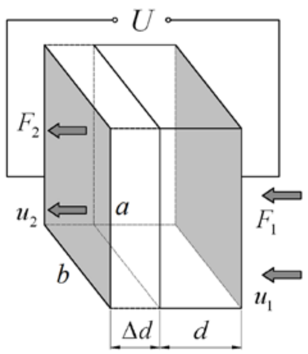

Fig. 1. The form of a single, free piezoelectric plate

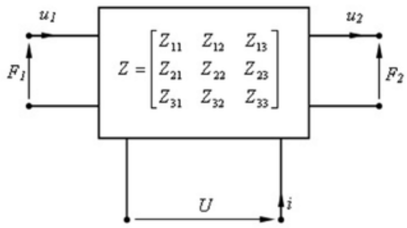

Fig. 2. A block diagram of a equivalent three-port system

Initially, in order to clarify the internal structure of the three-port system the value of a piezoelectric constant was assumed as zero. So, the system under consideration was not treated as piezoelectric and equations of forces acting on the plate can be described as [14]:

$$\begin{cases} F_1 = \dfrac{Z}{j\tan kd}\dot{u}_1 - \dfrac{Z}{j\sin kd}\dot{u}_2 \\ F_2 = \dfrac{Z}{j\sin kd}\dot{u}_1 - \dfrac{Z}{j\tan kd}\dot{u}_2 \end{cases}, \tag{1}$$

where:

$$k = \frac{\omega}{V}, \tag{2}$$

$$V = \sqrt{\frac{c}{\rho}}, \tag{3}$$

$$c = E + \frac{\varepsilon^2}{\varepsilon^S}, \tag{4}$$

$$Z = \rho V A, \tag{5}$$

In equations denotation was introduced: the symbol *j* denotes the imaginary unit, ε the relative deformation of the plate, ε^S the dielectric permittivity, ω a frequency of excitation, while symbols *E, A* and ρ denote the Young's modulus, the cross-section area and the density of the transducer [14].

The system of equations (1) describes dependences of the plate's surfaces movements on the applied forces. Using the electrical analogy a substitute electric circuit was introduced. It is a four port system of impedances connected in a star configuration. It is presented in Fig. 3.

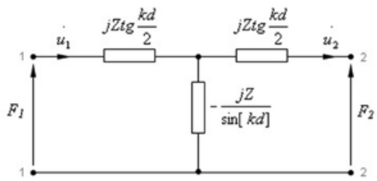

Fig. 3. A substitute electric circuit of the plate with the value of piezoelectric constant assumed as zero

Taking into account the non-zero value of the piezoelectric constant, the system of equations (1) can be supplemented by a relationship that describes the transformation of electrical energy into mechanical energy or in the opposite direction [14]:

$$hC_0\left(\dot{u}_1 - \dot{u}_2\right) = j\omega C_0 e_3 - i_3, \tag{6}$$

where:

$$h = \frac{\varepsilon}{\varepsilon^S}, \tag{7}$$

$$C_0 = \frac{\varepsilon^S A}{d}. \tag{8}$$

The left side of the equation (6) describes the current flowing in the longitudinal branch of the four port system after the transformation of mechanical values to electrical values. Using the transformation law, the considered system can be replaced by an ideal transformer with a ratio of:

$$r = hC_0. \tag{9}$$

The equivalent circuit of the piezoelectric plate that was created in this way is presented in Fig. 4.

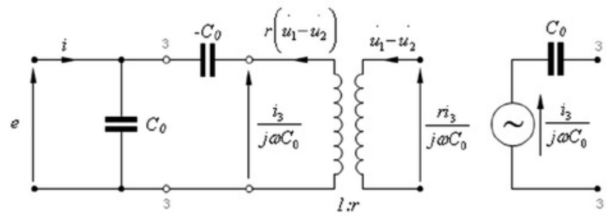

Fig. 4. A part of the electrical equivalent circuit of the piezoelectric plate

The transformation law is used for equivalent voltage, which in the Mason's equations are represented by an element [14]:

$$\frac{h}{j\omega}i = \frac{ri}{j\omega C_0}. \tag{10}$$

In the case of absence of piezoelectric coupling (r=0) there is not an electric current flow in the secondary winding of the transformer. So, the voltage of the electrical part can be described by the equation:

$$e = \frac{i}{j\omega C_0} . \tag{11}$$

In agreement with Thevenin theorem, the electrical current source was replaced by the voltage source and additional capacitance C_0 was introduced in parallel to the electrical input in the 3-3 node. The condition of equality of voltage on the electrical and mechanical part will be fulfilled in the case of joining a serial capacitor – C_0. The substitute Mason circuit of the piezoelectric plate that is powered by a parallel field is presented in Fig. 5. It represents mechanical and electrical parts.

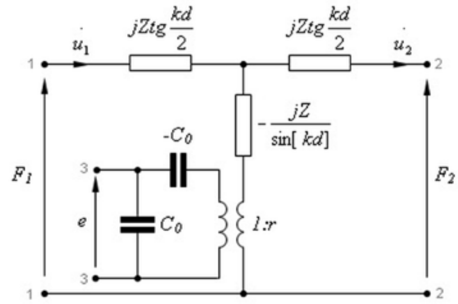

Fig. 5. The equivalent Mason circuit of piezoelectric plate

When using substitutive systems of piezoelectric transducers in the form of electrical circuits all mechanical values were replaced by an equivalent electrical values in the introduced Mason circuit:

$$e_1 = \frac{F_1}{r} , \quad e_2 = \frac{F_2}{r} , \tag{12}$$

$$i_1 = \dot{u}_1 r , \quad i_2 = \dot{u}_2 r , \tag{13}$$

$$Z_0 = \frac{Z}{r^2} , \tag{14}$$

$$r = \frac{eC_0}{\varepsilon^S} . \tag{15}$$

By substituting equations (12) to (15) in equations (1) and (6), the system can be described by equations:

$$e_1 = \frac{Z_0}{jtg[kd]} i_1 - \frac{Z_0}{j\sin[kd]} i_2 + \frac{1}{j\omega C_0} i_3 , \tag{16}$$

$$e_2 = \frac{Z_0}{j\sin[kd]} i_1 - \frac{Z_0}{jtg[kd]} i_2 + \frac{1}{j\omega C_0} i_3 , \tag{17}$$

$$e_3 = \frac{1}{j\omega C_0} i_1 - \frac{1}{j\omega C_0} i_2 + \frac{1}{j\omega C_0} i_3 , \tag{18}$$

and a corresponding Mason circuit can be presented as shown in Fig. 6.

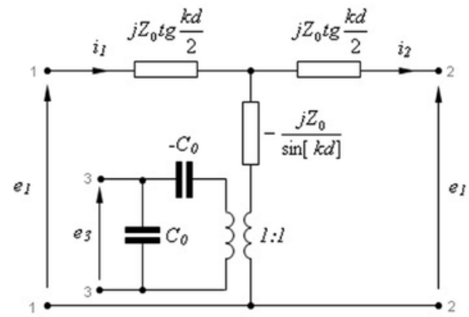

Fig. 6. The equivalent Mason circuit in electrical analogy of the piezoelectric transducer powered by the parallel field

The equivalent circuit of the piezoelectric transducer was transformed in such way that the forces F_1 and F_2 were replaced by electric voltages e_1 and e_2 and the values of displacement u_1 and u_2 by electrical currents i_1 and i_2. The ratio of turns of the primary winding to the number of turns of the secondary winding of the transformer is 1:1. Calculations carried out in the following part will therefore be conducted using the equivalent circuit shown in Fig. 6 and the corresponding equations. Note, however, that the arms 1-1 and 2-2 are representation of mechanical values.

3. The equivalent model of the piezoelectric transformer

The idea of creating a piezoelectric transformer appeared in the fifties of the twentieth century [15, 20, 23, 27]. Both, the direct and reverse piezoelectric effects are used in the piezoelectric transformer. By converting electrical energy into mechanical energy in the first piezoelectric plate of the transformer mechanical vibrations are being generated (the reverse piezoelectric effect). Vibrations are transmitted to the second piezoelectric plate and, as a result of mechanical deformation, electric charge is generated (simple piezoelectric effect). Operation of the piezoelectric transformer is illustrated in Fig. 7.

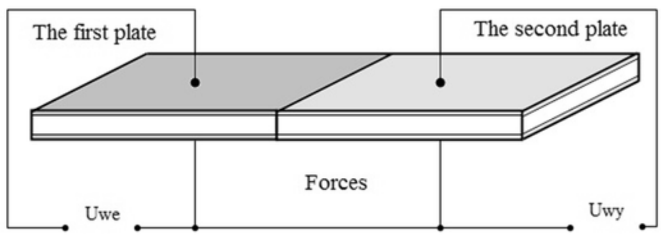

Fig. 7. Functional diagram of the piezoelectric transformer

In the presented case it was assumed that harmonic voltage is applied to the first piezoelectric plate of the transformer and causes vibration in the thickness direction of the plate. The second piezoelectric plate is used to recover and strengthening of the electric voltage. In the work [20] different types of piezoelectric transformers that transfer longitudinal vibrations are presented. Depending on the type of piezoelectric material used and the arrangement of electrodes the piezoelectric transformer is characterized by a specific gain value of the output voltage.

Modelling of the considered piezoelectric transformer by introduction of substitute, equivalent electrical circuits of each plate is presented in this paper. The first plate is subjected to external electric field and the other one to the forces generated by the first plate. Both plates are polarized in the direction of axis 3. Assumed equivalent circuit of a single piezoelectric plate is shown in Fig. 8 [23].

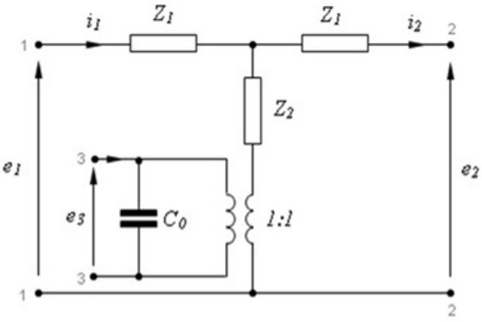

Fig. 8. Substitute electric circuit of a single, input piezoelectric plate of the transformer

Dependencies of impedances connected in a star configuration are described as:

$$Z_1 = Z_{1'} = jZ_0 tg\frac{[kd]}{2}, \tag{19}$$

$$Z_2 = Z_{2'} = \frac{Z_0}{j\sin[kd]}. \tag{20}$$

A diagram formed by connecting terminals of the pair of mechanical arms 2-2 is presented in Fig. 9. It is transformed model of the plate cooperating with the second piezoelectric transducer. In the place of the newly formed pair of arms 1-2 the second piezoelectric plate is coupled. The terminal 3-3 is powered by external input voltage U_{we} [23].

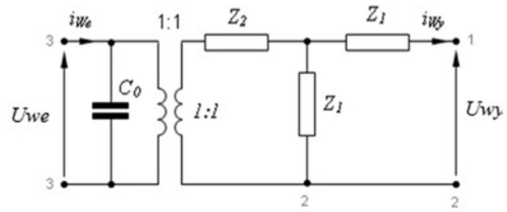

Fig. 9. The transformed substitute circuit of the input piezoelectric plate

In the case of the second piezoelectric plate its equivalent circuit is similar but supplemented by an additional capacitor C_0. An obtained equivalent circuit of the piezoelectric transformer is presented in Fig. 10.

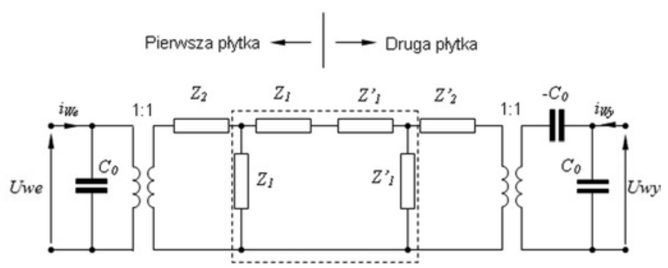

Fig. 10. The substitute circuit of the piezoelectric transformer transmitting longitudinal vibrations

The system consisting of the same type of piezoelectric plates was considered thus the piezoelectric transformer ratio, that is couplings between electrical and mechanical systems, is 1:1. The value of substitute impedance created from a connection of Z_1 and Z'_1 was determined:

$$Z_1{'}_1 = Z_1 + Z{'}_1 = 2jZ_0 tg\frac{[kd]}{2}, \tag{21}$$

and the connection of impedances Z_1, Z'_1 and $Z_1{'}_1$ was changed from the triangle configuration to the star configuration:

$$Z_A = \frac{Z_1 Z_1{'}_1}{Z_1 + Z{'}_1 + Z_1{'}_1} = \frac{\left(2jZ_0 tg\frac{[kd]}{2}\right)^2}{4jZ_0 tg\frac{[kd]}{2}} = \frac{j}{2}Z_0 tg\frac{[kd]}{2}, \tag{22}$$

$$Z_B = \frac{Z_1 Z_1{'}}{Z_1 + Z_1{'} + Z_1{'}_1} = \frac{\left(jZ_0 tg\frac{[kd]}{2}\right)^2}{4jZ_0 tg\frac{[kd]}{2}} = \frac{j}{4}Z_0 tg\frac{[kd]}{2}, \tag{23}$$

$$Z_C = \frac{Z_1{'} Z_1{'}_1}{Z_1 + Z_1{'} + Z_1{'}_1} = \frac{\left(2jZ_0 tg\frac{[kd]}{2}\right)^2}{4jZ_0 tg\frac{[kd]}{2}} = \frac{j}{2}Z_0 tg\frac{[kd]}{2}. \tag{24}$$

Obtained substitute circuit is presented in Fig. 11.

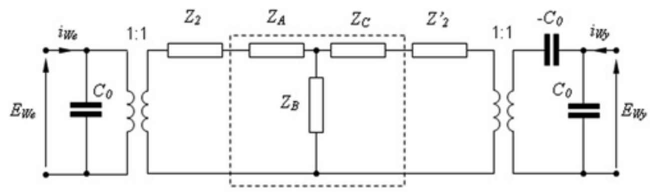

Fig. 11. Diagram of transformation of impedances connection from the triangle configuration to the star configuration

Further minimize of the number of system components was realized by combining elements respectively Z_2 together with Z_A and Z'_2 together with Z_C according to the equation:

$$Z_2 + Z_A = Z_C + Z_2{'} = \frac{Z_0}{j\sin[kd]} + \frac{jZ_0}{2}tg\frac{kd}{2}. \tag{25}$$

Using the trigonometric identity:

$$tg\frac{kd}{2} = \frac{1-\cos[kd]}{\sin[kd]}, \tag{26}$$

it was obtained:

$$Z_2 + Z_A = Z_C + Z_2' = \frac{Z_0}{j\sin[kd]} + \frac{jZ_0\left(1-\cos[kd]\right)}{2\sin[kd]} . \quad (27)$$

After transformations, equation (27) can be written in the form:

$$Z_2 + Z_A = Z_C + Z_{2'} = \frac{Z_0\left(1-\cos[kd]\right)}{2j\sin[kd]} . \quad (28)$$

Taking into account that:

$$tg\frac{kd}{2} = \frac{\sin[kd]}{1+\cos[kd]} , \quad (29)$$

it can be finally written down:

$$Z_2 + Z_A = Z_C + Z_2' = \frac{Z_0}{2j \cdot tg\frac{kd}{2}} . \quad (29)$$

The minimized form of the circuit is presented in Fig. 12.

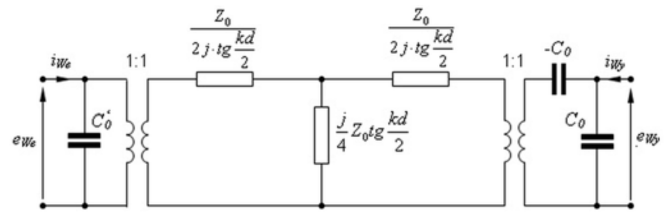

Fig. 12. Minimized circuit (containing a combination of resistors in a star configuration) of the piezoelectric transformer

Using electromechanical analogies the created system can be transformed to a resonant circuit composed of passive RLC components. This circuit is presented in Fig. 13.

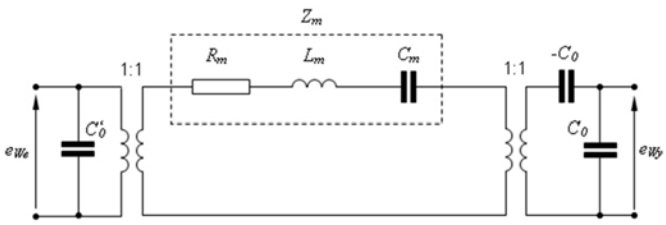

Fig. 13. Substitute scheme of the mechanical resonance system in the form of an electrical circuit

The obtained model is equivalent to a mechanical $R_mL_mC_m$ system [23]. The value of the first natural frequency can be described by the equation:

$$\omega_0 = \frac{\pi}{l}\sqrt{\frac{c}{\rho}} . \quad (30)$$

Values of the individual elements of passive resonant circuit were described by equations:

$$L_m = \frac{\pi Z_0}{4\omega_0} , \quad (31)$$

$$R_m = \frac{\pi Z_0}{4Q_m} , \quad (32)$$

$$C_m = \frac{1}{\omega_0{}^2 L_m} , \quad (33)$$

by symbol Q_m a quality constant was indicated [23].

Using equations (31) to (33) values of the individual elements of $R_mL_mC_m$ circuit were calculated. They were used to calculate the dynamic flexibility of the piezoelectric transformer without damping (in analogy of electrical admittance Y of the electric resonant circuit). The value of the Q_m constant was taken from [23] to be equal 2000. The obtained modulus of dynamic flexibility of the resonant system for the first natural frequency is shown in Fig. 14.

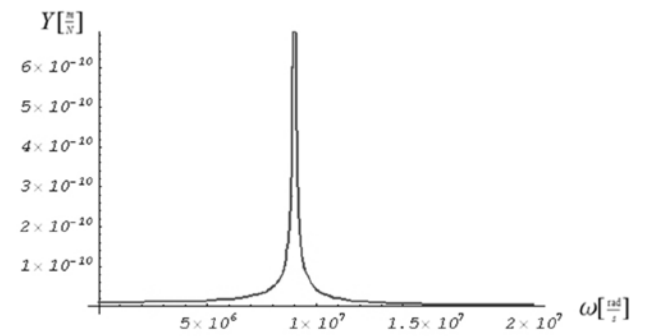

Fig. 14. Absolute value of the dynamic flexibility of mechanical resonant system

Fig. 14 shows designated course of absolute values of the dynamic flexibility of mechanical resonant system (the piezoelectric plate) obtained from the electrical analogy.

4. Conclusions

Presented algorithm can be used to develop an electrical equivalent circuit of a single piezoelectric plate and the piezoelectric transformer is an introduction to the process of modelling and testing of this type of vibrating systems that can be used to stabilize and damping of mechanical vibrations. In next works systems with vibrations damping will be considered. The damping will be introduced by connecting to the piezoelectric transducer passive electric circuits. The advantages of this type of passive vibration damping is primarily their low complexity and no need for an external power supply. This allows their applications in technical devices where is no possibility of access to the system during its operation or there is a need to reduce the complexity and energy consumption of the designed technical means.

Using the presented mathematical algorithm it is possible to realize modelling and testing of such systems, as well as analysis of the impact of their parameters on characteristics, including both parameters of the piezoelectric transducer as well as parameters of the external circuit. A necessary condition to realize a synthesis task, so designing of the system with required characteristics, is to develop a precise mathematical model of the designed system. Using this model it is possible to precise representation of the phenomena occurring in it.

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