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OPTIMISATION OF NON-PERIODIC INSPECTION AND MAINTENANCE FOR MULTICOMPONENT SYSTEMS

OPTYMALIZACJA NIE-OKRESOWYCH PRZEGLĄDÓW I KONSERWACJI SYSTEMÓW WIELOELEMENTOWYCH

A k-out-of-n:G system and a system with components subject to soft and hard failures are both inspected non-periodically. For the k-out-of-n system, components fail "silently" (i.e. are hidden), and the entire system fails when (n-k+1)st component fails. For the system with hard-type and soft-type components, hard failures cause system failure, while soft failures are hidden and do not cause immediate failure of the system, but still reduce its reliability. Every system failure allows for an opportunistic inspection of hidden soft-type components in addition to the scheduled inspections. The available maintenance types are replacement and minimal repair. For hard-type components, the maintenance decision is determined by the optimal age before replacement. For the soft-type components with hidden failures, we do not know their age, and so decide on the appropriate type of maintenance using the optimal number of minimal repairs before replacement. The hidden nature of soft-type component failures precludes the use of a tractable analytic expression, so we use simulation and genetic algorithm (GA) to jointly optimise the non-periodic policies on maintenance and inspection and to ensure these incur minimal expected total cost over a finite planning horizon. Due to increasing computational complexity associated with the number of inspections and maintenance policies to be evaluated, the genetic algorithm presents a promising method of optimisation for complex multicomponent systems with multiple decision parameters.

Keywords: *non-periodic inspection, opportunistic inspection, maintenance, hidden soft failure, hard failure, genetic algorithm.*

Przeglądów systemu typu k z n: G oraz systemu z elementami ulegającymi miękkim i twardym uszkodzeniom dokonuje się nieokresowo. W przypadku systemu k z n, składowe ulegają uszkodzeniom „w trybie cichym" (tj. uszkodzenia są ukryte), a cały system ulega awarii, gdy ulegnie uszkodzeniu element (n-k + 1). W przypadku systemu z elementami typu twardego i miękkiego, uszkodzenia twarde prowadzą do awarii systemu, natomiast uszkodzenia miękkie są ukryte i nie powodują natychmiastowej awarii systemu, choć nadal zmniejszają jego niezawodność. Każda awaria systemu stanowi dodatkową, w stosunku do przeglądów planowych, okazję do przeprowadzenia przeglądu (tzw. przegląd awaryjny) ukrytych elementów miękkich. Dostępne rodzaje konserwacji to wymiana oraz naprawa minimalna. W przypadku komponentów twardych, decyzję, który typ konserwacji zastosować, podejmuje się biorąc pod uwagę optymalny wiek przed wymianą. W przypadku elementów miękkich z ukrytymi uszkodzeniami, wiek optymalny jest nieznany, dlatego decyzje o odpowiednim typie konserwacji podejmuje się z uwzględnieniem optymalnej liczby minimalnych napraw przed wymianą. Ukryty charakter uszkodzeń elementów składowych typu miękkiego wyklucza wykorzystanie rozwiązywalnego wyrażenia analitycznego, dlatego w pracy użyto symulacji i algorytmu genetycznego (GA), w celu jednoczesnej optymalizacji nieokresowych strategii prowadzenia konserwacji i przeglądów oraz zapewnienia, że będą one pociągały za sobą minimalny oczekiwany koszt całkowity w skończonym horyzoncie planowania. W świetle rosnącej złożoności obliczeniowej związanej z dużą liczbą ocenianych przeglądów i strategii utrzymania ruchu, algorytm genetyczny stanowi obiecującą metodę optymalizacji złożonych systemów wieloelementowych o wielu parametrach decyzyjnych.

Słowa kluczowe: *przegląd nie-okresowy, przegląd awaryjny, utrzymanie ruchu, ukryte uszkodzenie miękkie, uszkodzenie twarde, algorytm genetyczny.*

Notation

Typesetting Convention: vectors, matrices and arrays are bold.

Latin Symbols

C	Cost (random variable).
E	Expectation.
F	Expected number of system failures.
G	Generator function for the expected values.
M	Expected number of minimal repairs.
P	Probability.
R	Expected number of replacements.
U	Expected uptime.
UCL	Upper confidence limit.
Y	First failure time for a soft-type component.
Z	First failure time for hard-type component subsystem.
a	Inspection policy index.
b	Random number.
c	Cost (constant).
f	Number of component failures.
h	Hard-type component index.
i	Scheduled inspection index.
j	Component index.

(*) Tekst artykułu w polskiej wersji językowej dostępny w elektronicznym wydaniu kwartalnika na stronie ***www.ein.org.pl***

k Number of identical components required to be in working order for a k-out-of-n system.
l Total number of inspections within a planning horizon.
m Number of minimal repairs before replacement.
n Total number of components.
s Soft-type component index.
t Age of a component.
x Inspection policy.
$\mathcal{R}$ Reliability.

Greek Symbols

Φ Random variable for the number of component failures.
Λ Cumulative hazard function.
Y Uptime of a component.
α Confidence level.
β Shape parameter in Weibull distribution.
ζ Optimal replacement age for hard-type component.
η Scale parameter in Weibull distribution, characteristic life.
θ Age of a hard-type component.
λ Power-law intensity (hazard) function.
σ Time until next inspection.
τ Minimal time unit.
φ Instance number of component failures.
χ Time-to-failure.
ω Length of planning horizon (life cycle).

Superscripts

* Optimality.
D Component downtime.
I Inspection.
M Minimal repair.
R Replacement (see "corrective replacement").
T Total.
HS System composed of hard-type and soft-type components.
SD System downtime.
a Inspection policy identifier.
k,n System composed of k-out-of-n components.

Subscripts

1 Reference to soft-type components.
2 Reference to hard-type components.
c Current.
h Hard-type component identifier.
i Scheduled inspection identifier.
j Component identifier.
l Reference to the total number of inspections within a planning horizon.
m Number of minimal repairs before replacement maintenance criterion.
s Soft-type component identifier.

1. Introduction and Background

Multicomponent systems generally have higher complexity than unicomponent systems, since the former usually have one or more intercomponent dependencies, such as functional, structural, failure, or economic [8, 28, 34]. Optimal maintenance and economic dependency in multicomponent systems is studied by Dekker et al. [11], Wang and Pham [36] and Zille et al. [41]. Periodic replacement policies for multicomponent systems with stochastic and economic dependencies are investigated by Ozekici [20]. Series systems with mixed standby components are compared based on their cost/benefit ratio, time to failure and long-term availability by Wang and Kuo [35].

Redundant systems with high levels of availability, reliability and robustness are typically configured as *k*-out-of-*n* systems, where the system is able to perform without interruption until failures of its components accumulate to *n*-*k*+1. Multi-engine aircraft, multi-display airplane cockpits, dual-contour automotive brake lines and multiple pumps used for hydraulic control are just several examples of *k*-out-of-*n* systems. A *k*-out-of-*n* system with perfect component repairs and maintenance equipment subject to imperfect repairs is considered by Zhang and Wu [38]. Load-sharing *k*-out-of-*n* systems are considered by Taghipour [27] and Taghipour and L. Kassaei [32]. They minimise the total expected cost and determine the optimal inspection interval for a finite planning horizon.

Definition 1.1: Generally, failure is an adverse event, which interferes with the normal designed functioning of the affected unit. One major class of multicomponent systems includes those composed of the two types of components classified by failure: hard-type and soft-type.

Definition 1.2: A *hard-type component* is a component whose failure is self-evident and triggers the system failure immediately; therefore, the time of failure is known for this component type. Examples of hard-type components include: wiring in ignition distributor in automotive electronic ignition, central processing unit in personal computers, fuse and display in infusion pumps, etc.

Definition 1.3: A *soft-type component* is a component whose failure does not trigger the immediate system failure, but the latter's reliability is usually reduced as a result of increased risk of malfunction, damage and/or eliminated redundancy. We refer to failures proper as "hard and soft failures" whenever the focus is on the failure process, and to components proper as "hard-type and soft-type components", respectively, to distinguish between different types and behaviours of components.

Definition 1.4: *System reliability* means the probability that the system will operate without failure under the design operating conditions (such as voltage, temperature, humidity). *Component reliability* refers to the same concept applied to individual components, whether hard-type, or soft-type.Examples of soft-type components include: liquid-level alarms in infusion pumps and standby-redundant components (batteries, surge-protective equipment, parallel processors).

Parts of the system subject to both soft and hard failures are treated as separate components of different types. It can be also noted that components in *k*-out-of-*n* systems can be treated as soft-type due to the system's capacity to accumulate component failures.

Periodic inspection policies for complex multicomponent systems have been extensively studied by Taghipour and Banjevic [31, 30, 29], Flage and Aven [12] and Pandey et al. [21]. Taghipour and Kassaei [32] consider periodic inspection optimisation for *k-out-of-n* systems.

For almost any system, the planning horizon is related to the system's life expectancy, depending on the operational and/or managerial objectives. Fixed and finite planning horizon is used in areas such as pharmacology, medical devices with expiry date, aircraft maintenance (Sriram and Haghani [25]). For example, medications and a vast majority of medical tools have to be replaced once the end of their life cycle has passed. Similarly, aircraft parts usually have to be preventively replaced after a specific number of flight hours.

Systems such as protective devices usually contain components whose failures are hidden. A hidden failure is a failure revealed only at inspection, but not during the normal operation of the system [19]. The detection of a hidden failure in an integrated system composed of main functional (protected) and safety (protective) units may occur either at inspection, or whenever the protective unit is required to function, but is unavailable because of a failure. Soft failures are similar to hidden failures, but the system is still able to function despite their presence. Single-component systems with hidden failures, probability of failure dependent on the number of previous repairs,

and maintenance policy based on both the component's age-at-failure and its number of overhauls are investigated by Sheu et al. [24].

Bjarnason et al. [6] consider a joint optimisation model for minimising the total cost of both maintenance and inventory policies for a system with hidden failures in a *k*-out-of-*n* redundant configuration. Babishin and Taghipour [2] propose a joint optimisation procedure for minimising the total cost of periodic inspections and additional inspections at system failures (opportunistic inspections), as well as corrective maintenance in *k*-out-of-*n* systems, where component failures are hidden.

Failure of an entire system or some of its components can be regarded as an opportunity to check all of the components for damage in addition to the scheduled inspections – hence, whenever such opportunity is taken, inspections performed at that time are called "opportunistic". In the literature, opportunistic maintenance has received an extensive treatment. For example, Dagpunar [10] considers a prespecified control limit for component's age, exceeding which a failed component in a multicomponent system is opportunistically replaced. Zhu et al. [40] offer a policy for opportunistic maintenance of offshore wind turbines with two component types, where maintenance action for soft-type components depends on their ages. Cui and Li [9] model damage in a multicomponent system accumulating to a shock event under opportunistic inspections and stochastically-dependent components. Aven and Dekker [1] consider age-based, as well as block replacement models with opportunities for preventive replacement. Gao et al. [13] propose a quasi-periodic imperfect preventive maintenance policy for a repairable system with stochastic maintenance interval. Peng et al. [23] study a sequential periodic preventive maintenance policy and develop a hybrid random imperfect maintenance model, optimising it using genetic algorithm. Legát et al. [18] consider both periodic preventive and predictive maintenance and determine, correspondingly, the optimal interval and the optimal diagnostic parameter. Gunn and Diallo [15] use a shortest path depth-first algorithm to search a network tree representation of the indirect opportunistic grouping of preventive periodic replacements. Yun and Endharta [37] use minimal cut set to analyse a *k*-out-of-*n*:F system with exponential failure times and evident failures.Unlike the cases from the literature, in the present case, there is a choice of maintenance action which the maintenance personnel may take at every failure of a component.

Genetic algorithms (GA) have been used in the literature for inspection optimisation of multicomponent systems. Because of the absence of analytical solution, Babishin and Taghipour [3] employ joint optimisation with exhaustive search, as well as genetic algorithm with either integer, or quasi-continuous inspection period. They provide the optimal joint inspection and maintenance policies, as well as calculate the expected number for system failures depending on the cost ratio and hazard function of components in a k-out-of-n system with hidden component failures under preventive or corrective replacement, minimal repair, opportunistic and periodic inspections. They also derive a criterion for calculating the acceptable number of system failures over its planning horizon. Bjarnason and Taghipour [5] formulate a model for a *k*-out-of-*n* system with a three-dimensional objective function and use the GA to find the joint optimal (s, S) inventory and maintenance policies. It should be noted that analysing systems with mixed hard and soft failures, where each component may require a special treatment, is generally much harder than analysing systems in *k*-out-of-*n* configuration with identical components.

Non-periodic inspection optimisation has also been covered in the literature. Zhao et al. [39] consider a system made of one component, which is replaced under a Gamma deterioration process over infinite planning horizon and develop a proportional hazards model for optimising the system's non-periodic inspection policy. Su [26] essentially develops a model for optimising non-periodic inspections in a one-component system with a combination of hidden and self-announcing operating modes, since his inspection "period" is a random variable, which renders it non-periodic according to the definitions and terminology adopted in the present paper. He uses the supplementary variable technique to optimise for an inspection period which maximises profit per unit time.

Multicomponent non-periodic inspections have also been considered in the literature. Hajipour and Taghipour's model [16] optimises for non-periodic inspection policy in a finite life cycle for multicomponent systems with a choice from two maintenance actions performed based on the age-dependent probability. Castanier et al. [7] propose a model taking into account the condition of the system for optimal inspection and replacement of a two-component system under non-periodic inspections, where they essentially develop separate policies for each component, assuming component independence, admitting that extending their approach to larger systems makes the numerical solution intractable. In this regard, it is worth mentioning that Vaurio notes in [33] that it is not generally possible to obtain an analytical solution for the optimal inspection interval even in the simpler case of optimising only for system availability. This explains the interest in and the value of numerical and simulation methods for the analysis of multicomponent systems.

Golmakani and Moakedi [14] develop a model for non-periodic inspection optimisation using branch-and-bound and dynamic programming techniques, which they use to introduce the A* search algorithm, which attempts to improve on the efficiency of branch-and-bound technique using branching on the most attractive nodes at each step in the procedure. However, the A*search is at a disadvantage for generating a large number of nodes at each iteration. Some researchers, e.g. Lapa et al. [17], demonstrated the applicability and usefulness of genetic algorithms to optimisation of system availability. In the present paper, genetic algorithm is used for the purpose of improving efficiency of optimisation calculations.

In summary, the present paper provides a general methodology and two models for finding the optimal joint non-periodic inspection and maintenance policies for complex multicomponent systems with finite planning horizon. In the previous models such as, for example, by Hajipour and Taghipour [16], Taghipour and Banjevic [31, 30, 29], the maintenance action was not optimised, and failed components were replaced, or minimally repaired based on age-dependent probability. Babishin and Taghipour [3]optimise both maintenance and inspection policies for a system in *k*-out-of-*n* configuration, but only under periodic inspections. Babishin and Taghipour [4] use a three-stage optimisation procedure to obtain optimal inspection policy for hard-type components in Stage 1, optimal maintenance in Stage 2 and optimal periodic inspection interval for soft-type components in Stage 3 using the Monte Carlo simulation.

In the present paper, both the maintenance decision and the inspection policy are optimised jointly in one stage. Recursive mathematical formulations for generating the expected values of minimal repairs, replacements and uptime are also provided for the first time in the case of a *k*-out-of-*n* system.The optimal maintenance policy for soft-type components is determined by the number of minimal repairs until replacement for these components, similarly to the approach proposed by Park [22]. The optimal maintenance policy for components with hard failures is based on these components' ages. Both of the proposed models feature corrective maintenance (replacement or minimal repair) of components with hard and soft failures, along with scheduled non-periodic and opportunistic inspections of components with soft failures. The hard failure occurrence in the system composed of hard- and soft-type components affects the expected number of soft failures, replacements, minimal repairs and expected downtime. Therefore, these expected values influence the optimal inspection policy. The components of a *k*-out-of-*n* system are regarded as being identical soft-type components, which facilitates the analysis of such systems. Jointly optimising for both inspection and maintenance in one stage for both systems allows finding optimal maintenance and

inspection policies for entire systems rather than marginally just for certain groups of components.

Generally, the safe and reliable operation of different equipment can be facilitated with the help of inspection and maintenance optimisation models. The latter also have strong managerial implications due to the importance of justifying these decisions with both qualitative and quantitative analysis. Using the proposed inspection and maintenance optimisation models, the decision-maker(s) gain an opportunity to find the combination of inspection and maintenance decisions that is most likely to result in the greatest cost savings without sacrificing availability or reliability. The hard-to-quantify effects, such as those of opportunistic inspections, can be accounted for by using the joint optimisation models in managerial decision-making process. This is likely to result in cost savings, which are especially significant, if the costs of inspection are high. Thus, it can be seen that optimisation of inspection and maintenance decisions represents a valuable asset for decision-makers.

The present article is further organised as following: Section 2 states the problem description; Section 3 outlines the model formulation for systems in *k*-out-of-*n* configurations under non-periodic and opportunistic inspections; Section 4 contains the model formulation for the system composed of hard-type and soft-type components under non-periodic inspections and opportunistic inspections of soft-type components; Section 5 illustrates the models by providing numerical examples; lastly, Section 6 summarises the conclusions.

2. Problem Description

Consider the problem of inspecting devices consisting of coupled systems, such as surge-protected personal computers (PCs), infusion pumps with liquid-level alarms, generators or power distributors with reserve power supplies. For such systems, it may not be economically feasible to have periodic inspections – for example, in the case when the optimal inspection period of the protective system does not coincide with the inspection period of the system they are coupled to. In such cases, non-periodic inspections are a good option.

In the present article, two main kinds of multicomponent systems are considered, based on the classification by the types of component failures. The system belonging to the first kind (System 1) is a *k*-out-of-*n* system with hidden component failures identifiable solely at inspections. The system belonging to the second kind (System 2) consists of components belonging to either of the two types: hard type, or hidden soft type. Both kinds of systems are considered in more detail in subsequent sections.

The present paper focuses on finding the optimal non-periodic policies for maintenance and inspection of two kinds of multicomponent systems described above. The relevant assumptions pertinent to System 1 are further identified by designation "S.1.#",those pertinent solely to System 2 – by "S.2.#", and those pertinent to both systems – by "S.1/2.#", where "S." stands for "system"). We start with stating the general assumptions for both kinds of systems:

S.1/2.1: Soft failures are discovered only at inspections. Therefore, the ages at failure of soft-type components are unknown.

S.1/2.2: Inspections are considered as being either scheduled non-periodic, or opportunistic.

S.1/2.3: Systems are always inspected at the end of the fixed planning horizon, all necessary maintenance is performed and components' ages recorded at that time in order to create a renewal point, after which the optimisation procedure can be repeated again.

S.1/2.4: Scheduled non-periodic (further referred to as simply "non-periodic") inspections occur with the minimal unit of time over a finite planning horizon ω, possibly at times $i\tau$, i=1,2,…,l, $l\in\mathbb{N}$, where $l=\omega/\tau$-1 if ω is divisible by τ, and $l=|\omega/\tau|$ otherwise. Scheduled inspections are always performed on the operating (unfailed) system at times prescribed by an inspection policy.

S.1/2.5: System failures offer an occasion to inspect every component in a system. Every failed component is then maintained to restore its functionality.

S.1/2.6: A maintenance action is classified as either a minimal repair, or a corrective replacement (further referred to as simply "replacement"). A minimal repair restores the component's functionality to the state it was in just preceding the component's failure, thus leaving the component's age unaffected. A corrective replacement decreases the failed component's age to 0 ("as-good-as-new" state). Minimal repairs and replacements can take place at scheduled, as well as opportunistic inspections.

S.1/2.7: Both maintenance and inspection are assumed to have negligible duration and are perfect.

When obtaining the total expected cost for a number of minimal repairs before replacement m_j per component within a given planning horizon, we assume the following:

S.1/2.8: Component j is replaced after m_j minimal repairs.

Furthermore, for the purposes of calculating the upper bound for the expected number of component failures within the planning horizon at α confidence level for each soft-type component, we assume that:

S.1/2.9: There is no delay in detecting failures upon inspection.

S.1/2.10: Component failures are rectified by minimal repair. (No replacements are considered here, since we are interested in finding the optimal number of times a failed component may be minimally repaired before it is replaced for the first time).

The number of minimal repairs before replacement depends on the expected number of component failures, since there is a statistical uncertainty associated with the latter. In obtaining the expected number of component failures $E[\Phi(\omega)]$, we observe the following about the component failure process:

S.1/2.11: $\Phi(t)\in\{\mathbb{N}\cup\{0\}\}$, (i.e. the number of failures is a non-negative integer).

S.1/2.12: If $t_1<t_2$, then $\Phi(t_1)\leq\Phi(t_2)$.

S.1/2.13: For $t_1<t_2$, $\Phi(t_2)-\Phi(t_1)$ equals the number of failures which occurred in the interval (t_1,t_2). Without the loss of generality, if t_1=0, t_2=ω, then $\Phi(t_2)-\Phi(t_1)=\varphi$.

Based on S.1/2.11-S.1/2.13, it can be concluded that the component failure process is a counting process. Furthermore:

S.1/2.14: $\Phi(0)$=0, (i.e. no components have failed prior to the beginning of the system's life cycle).

S.1/2.15: Component failures follow independent increments, i.e. the numbers of component failures in disjoint time intervals are mutually independent of each other.

For System 1, the following assumptions are also made:

S.1.1: The number of components for System 1 is denoted as n.

S.1.2: All redundant components in *k*-out-of-*n* configuration are identical.

S.1.3: Components' failures are assumed to occur according to a power law intensity function (hazard function) $\lambda_j(t)=\frac{\beta_j}{\eta_j}\left(\frac{t_j}{\eta_j}\right)^{\beta_j-1}$ following non-homogeneous Poisson process (NHPP), where β_j is the shape parameter and η_j is the scale parameter of the Weibull distribution describing times between failures of component j, t_j is the age of component j, j=1,2,…,n for System 1.

S.1.4: Opportunistic inspections are incurred whenever n-k+1 components fail.

S.1.5: The (n-k+1)th failure presents an opportunity for inspecting the system and rectifying the failed components, which influences the number of replacements, minimal repairs and downtime for the hidden components (see Fig.1).

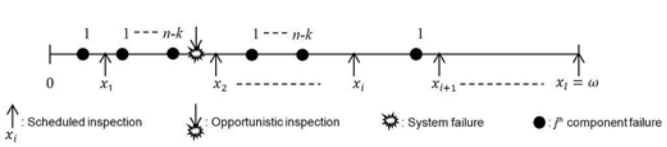

Fig.1: Sample hidden failures, scheduled and opportunistic inspections within one life cycle for System 1.

Consider the problem of optimising inspection and maintenance of several identical antennae providing network access on a remote base station (e.g. used in geophysical surveys), which because of crew staffing shortages, weather, or accessibility issues requires non-periodic inspection and maintenance. Each antenna is considered as a component, and the collection of antennae providing network access are considered as a system. When an antenna's battery runs down, it ceases receiving and retransmitting the signal, which decreases the overall signal coverage and signal strength. Because the signal has to be accessible from a helicopter, which can be flying anywhere within the coverage area, failure of one or more of the antennae constitutes a decrease in the performance of the entire system. The base station system is modelled as a k-out-of-n system, referred to as System 1. Fig. 1 provides an example of a k-out-of-n system, checked at scheduled non-periodic inspections (denoted by x_i) and opportunistically whenever n-k+1 components fail. The numbers 1,…,n-k above the black circles are denoting the ordinal number of component failures in the system over the time between inspections (and, hence, between failure rectifications). This is used to demonstrate an example of possible occurrence and accumulation of failures within a certain period of time.

Similarly to the above-stated assumptions for System 1, the following assumptions are made for System 2:

S.2.1: The number of soft-type components in System 2 is denoted as n_1, and the number of hard-type components is represented as n_2.

S.2.2: Similar to S.1.3, but with j=1,2,…,n_1+n_2 for System 2, where s=1,2,…,n_1 is the number of soft-type components and h=1,2,…,n_2 is the number of hard-type components in the system.

S.2.3: Opportunistic inspections are incurred whenever a hard failure occurs.

S.2.4: Hard failures create more opportunities for inspecting soft-type components and, consequently, influence the replacement and minimal repair numbers, as well as the downtime of components with soft failures (see Fig. 2).

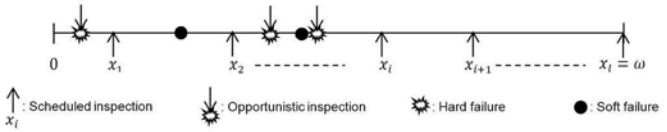

Fig.2: Sample hard and hidden soft failures, scheduled and opportunistic inspections within one life cycle for System 2.

Another problem involves optimising non-periodic inspection and maintenance policies for a MacPherson-type strut assembly found in automotive vehicles. Here, a shock absorber, a coil spring and a strut-to-mount nut are modelled as hard-type components, and lower and upper spring insulators, dust shield, jounce bumper and spring seat pad are modelled as soft-type components. The entire assembly constitutes a system and is referred to as System 2. Similarly to Fig. 1, Fig. 2 shows an example of hard and soft failures along with the scheduled and opportunistic inspections for System 2.

Once the end of the planning horizon (i.e. time ω) is reached, a new non-periodic optimal policy for maintenance and inspection can be established by repeating the outlined procedure. In the case of System 2, current ages of hard-type components can be taken into account when planning for the system life cycle.

The total cost of system maintenance and inspection is a metric used almost universally in different areas of industry for a large variety of systems. It is a convenient measure of the optimality of a system, since the latter's reliability and availability are connected through a range of costs, such as inspection and maintenance costs, component and system downtime penalties. For this reason, for both System 1 and System 2, the objective function is formulated based on the total cost of joint maintenance and inspection policy.

A closed deterministic formulation requires knowledge of all system parameters with certainty. However, this condition is not satisfied, because the failure ages of hidden soft-type components are unavailable. Instead, an expression is formulated to recursively find the expected system parameters (please refer to Appendix for details).

However, the recursive formula cannot be solved analytically because of some terms having multidimensional integrals requiring discretisation, which makes the computations cumbersome. Because of this, the present analysis is based on the results obtained from Monte Carlo simulations, as well as on the use of the genetic algorithm (GA).

To summarise, our objective is to find the non-periodic optimal maintenance policy m^* for System 1 and m_s^* for System 2, and optimal inspection policy x^*. The optimal joint policies are achieved through minimising for the whole system the total expected cost within the system's life cycle ω.

3. Model 1: joint optimisation of non-periodic inspection and corrective maintenance of *k*-out-of-*n* system with opportunistic inspections

In this section, we propose a model for a k-out-of-n system which may be non-periodically inspected at potential times $i\tau$, where τ is the minimal time unit. At the same time, maintenance optimisation is done for the discrete-valued number of component's minimal repairs until replacement. Overall, joint quasi-continuous and discrete optimisation is performed to obtain the joint optimal inspection and maintenance policies.

Maintenance optimisation is concerned with finding the best maintenance action in a particular system setting. In this paper, at each inspection point, the decision has to be made whether to minimally repair, or replace the failed component(s). Inspection optimisation then provides the best points in time at which these maintenance actions have to be taken in order to incur the lowest cost. Since failures are stochastic in nature, the total expected cost is used for optimality computations.

The scheduled non-periodic inspection policy $\boldsymbol{x}^a = (x_1, x_2, \ldots, x_l)$, where a refers to the inspection policy index, can be encoded as a binary sequence of '1's and '0's, where each '1' corresponds to a scheduled inspection and '0' corresponds to the lack thereof. Taking the number of digits of $\boldsymbol{x}^a$ to be l, each digit then corresponds to time $i\tau$. This binary representation lends itself naturally to the encoded "genome" strings used in the genetic algorithms, which makes it particularly convenient and effective for the purposes of inspection optimisation using the latter. The total number of possible distinct sched-

uled inspection policies is then 2^{l-1}, since there is always 1 inspection scheduled to occur at time ω. Hence, enumerating $\boldsymbol{x}^a$, $a=1,2,\ldots,2^{l-1}$.

The components' failures are hidden, making their ages at failure unknown. For this reason, maintenance decisions cannot be age-based. Instead, the number of minimal repairs is counted for each component, and the decision of whether to minimally repair, or replace a failed component is based on the number of minimal repairs until replacement. Since all components are assumed to be identical in *k*-out-of-*n* configuration, only one optimal number of minimal repairs before replacement has to be found for a given system.

Based on observations S.1/2.11-S.1/2.13, it can be concluded that the component failure process is a counting process. Moreover, observations S.1/2.14-S.1/2.15 pertain to a Poisson process, and assumptions S.1.3 and S.2.2 further specify the sequence of random variables $\{\Phi(t)\}_{t\geq 0}$ describing the failure process as NHPP. Making use of assumptions S.1/2.9 and S.1/2.10,the expected number of failures $E[\Phi(\omega)]$ to time $\omega \geq 0$ is obtained as following:

$$E[\Phi(\omega)] = \Lambda(\omega) = \int_0^{\omega} \lambda(t)dt = \int_0^{\omega} \frac{\beta}{\eta}\left(\frac{t}{\eta}\right)^{\beta-1} dt = \left(\frac{\omega}{\eta}\right)^{\beta}, \quad (1.1)$$

where $\Lambda(\omega)$ is a cumulative hazard function.

The actual number of failures is expected to fall within a $100*\alpha$ % confidence interval, with the upper confidence limit *UCL* given from Poisson distribution as:

$$UCL = \min\left\{\varphi \in \mathbb{N}_0 : P(\Phi(\omega) \leq \varphi) \geq 1-\frac{1-\alpha}{2}\right\}$$
$$= \min\left\{\varphi \in \mathbb{N}_0 : \sum_{f=0}^{\varphi} P(\Phi(\omega) = f) \geq 1-\frac{1-\alpha}{2}\right\}, \quad (2.1)$$

where *UCL* is the upper confidence limit for a component and is dependent on ω, and the probability of observing φ failures over planning horizon ω is given by $P(\Phi(\omega)=\varphi) = \frac{\left(E[\Phi(\omega)]\right)^{\varphi}}{\varphi!}\exp\left(-E[\Phi(\omega)]\right)$ for each soft-type component.

The total expected cost $E\left[C^{T,k,n}_{\boldsymbol{x}^a,m}\right]$ is formulated as following:

$$E\left[C^{T,k,n}_{\boldsymbol{x}^a,m}\right] = lc^I + F(\omega,\boldsymbol{t},f_s,m_c)c^{SD}$$
$$+\sum_{j=1}^{n}\left(c^M M_{\boldsymbol{x}^a,m,j}(\omega,t_j,f_s,m_c)\right.$$
$$+c^R R_{\boldsymbol{x}^a,m,j}(\omega,t_j,f_s,m_c)$$
$$\left.+c^D\left(\omega - U_{\boldsymbol{x}^a,m,j}(\omega,t_j,f_s,m_c)\right)\right), \quad (3)$$

where *l* is a number of scheduled inspections, c^I is a cost of a scheduled inspection, $\boldsymbol{t}=(t_1,t_2,\ldots,t_n)$ is a vector with initial ages of components, $F(\omega,\boldsymbol{t},k,n)$ is the expected number of failures for the system, c^{SD} is a downtime penalty for the system, c^D is a downtime cost per component per unit time, c^R is a per-component cost of corrective replacement, c^M is a per-component cost of minimal repair, t_j is an initial age of component j, m is a number of minimal repairs until replacement, $U_{\boldsymbol{x}^a,m,j}(\omega,t_j,f_s,m_c)$, $R_{\boldsymbol{x}^a,m,j}(\omega,t_j,f_s,m_c)$ and $M_{\boldsymbol{x}^a,m,j}(\omega,t_j,f_s,m_c)$ are, respectively, the expected uptime and the expected numbers of replacements and minimal repairs for each component. In the proposed formulation, expected values are generally obtained recursively using the generator function $G_{x_i,m_s,s}(\omega,t_s,f_s,m_c)$ with the variables indicated inside the brackets as parameters (see Appendix for details).

The results of the periodic optimisation from both exhaustive search and genetic algorithm search procedures were cross-verified and were found to be identical. Using the same logic and modifying the code to accommodate non-periodic frequency of inspections, we extrapolate the results to the non-periodic domain.

The optimal joint maintenance and inspection policies are determined by the optimal inspection policy $\boldsymbol{x}^*$ and the optimal number of minimal repairs until replacement m^*, respectively. Using calculations for the combinations of possible inspection and maintenance policies ($\boldsymbol{x}^a, m$), the optimal joint inspection policy ($\boldsymbol{x}^*, m^*$) can be obtained from searching for the smallest total expected cost as following:

$$(\boldsymbol{x}^*, m^*) = \min_{\boldsymbol{x}^a,m}\left\{E\left[C^{T,k,n}_{\boldsymbol{x}^a.m}\right]\right\},$$
$$\text{s. t.}: 0 \leq m \leq UCL,$$
$$x_i = \begin{cases} 1, \textit{if inspection occurs at time } i\tau \\ 0, \textit{if no inspection occurs at time } i\tau \end{cases},$$
$$i = 1,2,\ldots,l. \quad (4)$$

The expected values required for the calculation of $E\left[C^{T,k,n}_{\boldsymbol{x}^a.m}\right]$ are, however, unavailable for systems where some or all of components fail in hidden mode, because failure ages of these components are unavailable and cannot be formulated explicitly. This obstacle is overcome by using the simulation procedures described in Section 5.

4. Model 2: non-periodic inspection and corrective maintenance of hard-type and soft-type components with opportunistic inspection of soft-type components

This section describes the methodology for finding the optimal maintenance actions after failures and the optimal inspection policy for System 2, taking into account the fact that soft failures are hidden and the soft-type components' ages at the time of failure are unknown. The model resulting from this methodology is called "Model 2" further in the text.

Due to the different failure characteristics, hard-type components are analysed separately from the soft-type components. Hard failure times are known, since the system stops operating immediately when-

ever a hard failure occurs. The goal is to determine the optimal ages at which the hard-type components should be replaced, providing the lowest cost of inspection and maintenance for the entire system. In order to achieve this, the domain of possible replacement ages from which to choose the optimal ones has to be defined for each hard-type component. The replacement ages are represented by vector $\boldsymbol{\zeta}=\left(\zeta_1,\zeta_2,\ldots,\zeta_{n_2}\right)$, consisting of replacement ages for each hard-type component h=1,2,…,n_2 .

From the system life cycle's perspective, it is impractical to make the hard-type component's replacement age longer than the life span of the entire system, as represented by its planning horizon. It is assumed that an overhaul or similar renewal event is to take place at the end of the system's life cycle, at which point those hard-type components which have not been maintained over the system's operation will be replaced. Thus, the replacement ages for hard-type components can be assumed to be bounded by 0 from the bottom and a multiple of the system's life cycle length at the top, for example: $0<\zeta_h\leq 1.5\omega$, h=1,2,…,n_2 .The choice of the multiple of ω is arbitrary and depends on the practical considerations rather than the theoretical ones. The motivation for choosing 1.5 as a multiple for the upper bounds is to allow optimal replacement ages vary in the range greater than the system's life cycle length for greater generality, but at the same time not to waste computational resources checking for unrealistically long replacement ages that are impractical for planning purposes.

The optimal replacement ages for all hard-type components are represented by vector $\boldsymbol{\zeta}^*=(\zeta_1^*,\zeta_2^*,\ldots,\zeta_n^*)$ consisting of the optimal replacement ages for each hard-type component h=1,2,…,n_2.

Unlike those for hard failures, the soft failure times are unknown, which makes it impossible to base the optimisation procedure on the ages of soft-type components. Instead, maintenance decision can be based on the number of minimal repairs until replacement. Similarly to System 1 and using the same assumptions, the expected number of failures $E[\Phi_s(\omega)]$ for System 2 was obtained as following:

$$E\left[\Phi_s(\omega)\right]=\int_0^{\omega}\frac{\beta_s}{\eta_s}\left(\frac{t_s}{\eta_s}\right)^{\beta_s-1}dt=\left(\frac{\omega}{\eta_s}\right)^{\beta_s}. \tag{1.2}$$

The actual number of component failures, however, may vary, owing to the stochastic nature of component failures. Hard failures are assumed to be rectified immediately upon failure. Soft failures are rectified at the earlier of either a scheduled inspection, or hard failure (i.e. at opportunistic inspection). We may then get the general estimate on the upper bound of the number of minimal repairs until replacement from using Poisson distribution for $E[\Phi_s(\omega)]$ to construct a confidence interval at α level as following:

$$UCL_s=\min\left\{\varphi_s\in\mathbb{N}_0:\sum_{f_s=0}^{\varphi_s}P\left(\Phi_s(\omega)=f_s\right)\geq 1-\frac{1-\alpha}{2}\right\}, \tag{2.2}$$

where UCL_s is the upper confidence limit for soft-type component s and the rest of terms are as previously defined.

We assume that the number of minimal repairs before replacement m_S for soft-type component s, s=1,…,n_1 does not exceed the upper confidence limit UCL_s on the mean of component failures and may take on any value between 0 and UCL_s, inclusively. Thus, different cases are covered, ranging from replacing component at every failure to replacing it at (UCL_s+1)st failure, with m_C keeping track of the current number of minimal repairs. Furthermore, m_S thus selected serves as the criterion for making a maintenance decision. Component s is minimally repaired at each inspection so long as no more than m_S failures occur. It is then replaced on (m_S+1)st failure. The optimal number of minimal repairs until replacement m_s^* results in the lowest total expected cost $E\left[C^{T,HS}_{\boldsymbol{x}^a,m_s,\zeta_h}\right]$ for the entire system.

It should be noted, that unlike in preventive replacement models for mixed systems composed of hard- and soft-type components encountered in Babishin and Taghipour [4], corrective replacement models may exclude the costs of hard-type components from the optimisation. This is because the hard-type components are replaced at the optimal replacement ages if they fail, and not at the scheduled inspection times when they are still operational. This makes the optimal non-periodic inspection independent of the costs of hard failures.

In order to obtain the lowest expected cost, all combinations of non-periodic inspection schedules, the numbers of minimal repairs before replacement, and various ages as threshold for replacement have to be considered for all components. The expected costs thus calculated can then be searched for the lowest value. However, the size of the search space is very large in this case. For this reason, this problem, albeit in the context of periodic inspections, has been previously broken down into several stages for maintenance and inspection optimisation for all hard-type components in Stage 1, marginal optimisation of the maintenance decision for each soft-type component in Stage 2 and optimisation of the inspection period for the entire system in Stage 3, using Monte Carlo simulation for marginal multi-stage optimisation [4].

In the present paper, global system-level optimisation is performed, which requires simultaneous optimisation of all decision variables. This results in a dramatic increase of the search space. The latter is greatly reduced by means of the genetic algorithm. This allows optimising for both inspection and maintenance jointly in one stage. The total expected cost $E\left[C^{T,HS}_{\boldsymbol{x}^a,m_s,\zeta_h}\right]$ is calculated as following:

$$\begin{aligned}E\left[C^{T,HS}_{\boldsymbol{x}^a,m_s,\zeta_h}\right]=lc^I+\sum_{s=1}^{n_1}\Big(&c_s^M M_{\boldsymbol{x}^a,m_s,s}(\omega,t_s,\boldsymbol{\theta},f_s,\boldsymbol{\zeta},m_c)\\&+c_s^R R_{\boldsymbol{x}^a,m_s,s}(\omega,t_s,\boldsymbol{\theta},f_s,\boldsymbol{\zeta},m_c)\\&+c_s^D\left(\omega-U_{\boldsymbol{x}^a,m_s,s}(\omega,t_s,\boldsymbol{\theta},f_s,\boldsymbol{\zeta},m_c)\right)\Big)\end{aligned} \tag{5}$$

where superscript HS indicates the cost for System 2 consisting of components with both soft and hard failures, $\theta=(\theta_1,\theta_2,\ldots,\theta_h,\ldots,\theta_{n2})$ is a vector containing the initial ages of hard-type component h, h=1,2,…,n_2, $\boldsymbol{\zeta}=\left(\zeta_1,\zeta_2,\ldots,\zeta_{n_2}\right)$ is a vector with replacement ages of hard-type components, t_S is the initial age of component s, c_s^M is a minimal repair cost of component s, c_s^R is a replacement cost for each soft-type component, c_s^D is a cost of downtime for component s, m_C is the current number of minimal repairs, and terms $U_{\boldsymbol{x}^a,m_s,s}(\omega,t_s,\boldsymbol{\theta},f_s,\boldsymbol{\zeta},m_c)$, $R_{\boldsymbol{x}^a,m_s,s}(\omega,t_s,\boldsymbol{\theta},f_s,\boldsymbol{\zeta},m_c)$ and $M_{\boldsymbol{x}^a,m_s,s}(\omega,t_s,\boldsymbol{\theta},f_s,\boldsymbol{\zeta},m_c)$ represent, respectively, the expected uptime and the expected numbers of replacements and minimal repairs for each soft-type component s, s=1,2,…,n_1.

Using the calculations for the combinations of possible inspection and maintenance policies $\left(\boldsymbol{x}^a,m_s,\zeta_h\right)$, the optimal joint inspection and maintenance policy $\left(\boldsymbol{x}^*,m_s^*,\boldsymbol{\zeta}^*\right)$ can be obtained from searching for the smallest total expected cost as following:

$$\left(\boldsymbol{x}^*, m_s^*, \zeta^*\right) = \min_{\boldsymbol{x}^a, m_s, \zeta_h} \left\{ E\left[C^{T,HS}_{\boldsymbol{x}^a, m_s, \zeta_h} \right] \right\},$$
$$\text{s. t.}: 0 \le m_s \le UCL_s,$$
$$s = 1,2,\ldots,n_1,$$
$$0 \le \theta_h \le \zeta_h,$$
$$h = 1,2,\ldots,n_2,$$
$$x_i^a = \begin{cases} 1, \text{ if an inspection occurs at time } i\tau \\ 0, \text{ if no inspection occurs at time } i\tau \end{cases}$$
$$i = 1,2,\ldots,l,$$
$$a = 1,2,\ldots,2^{l-1}. \quad (6)$$

The following section outlines the general simulation procedure used for optimisation.

5. Simulation Model

Simulation procedure is similar for both systems, but differs in some details as a result of the difference in the types of system's components.

5.1. Simulation Model for *k*-out-of-*n* System (System 1)

The simulation for the *k*-out-of-*n* system takes as inputs the values of x_i^a, m, k, n, ω, τ, t_j, β, η, c^M, c^{CR}, c^{PR}, c^D, c^I and c^{SD}. Let the random variable Y_j (uptime of component j) have a Weibull distribution with parameters β and η. This component has an age t_j and probability that the time-to-failure is equal to χ_j, which is given by formula:

$$P\left(\mathrm{Y}_j = \chi_j + t_j | \mathrm{Y}_j \ge t_j\right) = \frac{\mathcal{R}\left(\chi_j + t_j, \beta, \eta\right)}{\mathcal{R}\left(t_j, \beta, \eta\right)}, \quad (7)$$

where $\mathcal{R}\left(t_j, \beta, \eta\right)$ is a reliability function. To generate the time-to-failure for component j, $j = 1,2,\ldots,n$, we first generate a random number b, which has a uniform distribution on interval $[0;1]$, and next calculate a quantile of order b for conditional distribution in Equation (7).The time-to-failure χ_j is generated as following:

$$\chi_j = \eta\left(\left(\frac{t_j}{\eta}\right)^{\beta} - \ln(b)\right)^{1/\beta} - t_j. \quad (8)$$

Generated times-to-failure are then compared with the time $i\tau$ of the earliest scheduled inspection flagged as '1' in $\boldsymbol{x}$. While $\chi_j < i\tau$, the number of failures of component j is increased by 1. Once total component failures in the system accumulate to $n-k+1$ failures, failure of entire system occurs, giving rise to opportunistic inspection, during which all failures are discovered. A failed component is minimally repaired if it has failed for m times or fewer; alternatively, it is replaced, and the failure count for it is reduced to zero.

While $\chi_j < i\tau$, but there are fewer than $n-k+1$ failed components, the latter are fixed at the following scheduled inspection. Again, a failed component is minimally repaired if it has failed for m times or fewer; alternatively, it is replaced, and the failure count for it is reduced to zero.

While $\chi_j > i\tau$, the simulation clock is moved forward to the inspection time, since there is no failed component to be discovered at inspection.

The simulation clock is updated at the times of events, such as component failures, system failures, and scheduled and opportunistic inspections. The downtime of component j, the number of system failures, the number of minimal repairs and replacements are all updated at each event's time as well. At the same time, the ages of the surviving components, the time until the next failure, and the time until the scheduled inspection are also updated. The simulation stops when the system's life cycle is completed. Running the simulation for a large number of times provides the expected values of the random variables F, $M_{\boldsymbol{x}^a,m,j}$, $R_{\boldsymbol{x}^a,m,j}$ and $U_{\boldsymbol{x}^a,m,j}$.

A given policy ($\boldsymbol{x}^a, m$) prescribes the choice of the maintenance action at each simulation run. Varying the values of $\left(\boldsymbol{x}^a, m\right)$ in the ranges $\boldsymbol{x}^a = (0,0,\ldots,0),(1,0,\ldots,0),\ldots,(1,1,\ldots,1)$, $0 \le m \le UCL$, the total expected cost $E\left[C^{T,k/n}_{\boldsymbol{x}.m}\right]$ for policy $\left(\boldsymbol{x}^a, m\right)$ is computed and saved. Thus, the total number of distinct policies in the decision space for Model 1 is $2^{\omega/\tau - 1} * UCL$.

Lastly, the joint optimal inspection and maintenance policy $\left(\boldsymbol{x}^*, m^*\right)$ is found from searching for the minimum $E\left[C^{T,k,n}_{\boldsymbol{x}^*.m^*}\right]$.

5.2. Simulation model for a system with hard-type and soft-type components (System 2)

The general simulation procedure for System 2 is similar to that described for System 1. The following input variables are used: x_i^a, m_c, m_s, ω, τ, t_s, $\theta_h, \beta_s, \beta_h, \eta_s, \eta_h, \zeta_h$, c_s^M, c_h^D, c_s^R, c_h^D, c_s^D and c^I. The same procedure as discussed in the previous section is used to generate the times for events and update the simulation clock.

If $\chi_s < \chi_h < i\tau$ for the generated soft failure time χ_s, the closest hard failure time χ_h and the closest non-periodic inspection time $i\tau$, then an opportunistic inspection is occurring at the closest hard-type component's failure time. The soft-type component's failure is detected at this moment, and if the total number of previous failures is less than m_s, the component is minimally repaired; otherwise, it is replaced, and its failure count is reset to zero.

When $\chi_h \le \chi_s < i\tau$, soft failure is fixed at the soonest scheduled inspection.

If $\chi_h < i\tau$, the age of the failed hard-type component is compared with its corresponding replacement age ζ_h, and the hard-type component is replaced if $\theta_h > \zeta_h$, or it is minimally repaired otherwise. The component's age is set to zero at replacement.

Changing the values of a given joint inspection and maintenance policy $(\boldsymbol{x}^a, m_s, \zeta_h)$ in the ranges $\boldsymbol{x}^a = (0,0,\ldots,0),(1,0,\ldots,0),\ldots,(1,1,\ldots,1)$, $0 \le m_s \le UCL_s$, $s = 1,2,\ldots,n_1$, $0 \le \zeta_h \le 1.5\omega$, $h = 1,2,\ldots,n_2$, the total expected cost for each policy is computed and saved. Thus, the total number of distinct policies in the decision space for Model 2 is $2^{\omega/\tau-1} * UCL_{n_1}{}^{n_1} * (1.5\omega/\tau)^{n_2}$.

Finally, the optimal solution is the one with the minimum cost $E\left[C^{T,HS}_{\boldsymbol{x}^*,m_s^*,\zeta_h^*}\right]$ over all input variables, i.e. inspection policies, possible values of the number of minimal repairs before replacement for soft-type components and possible values for the optimal replacement ages for hard-type components. The resulting triple $\left(\boldsymbol{x}^*, m_s^*, \zeta^*\right)$ represents the optimal joint inspection and maintenance policy.

In using simulation as described above, however, there is a significant drawback related to the search method used for optimisation. The search method for the optimal joint policy is highly sensitive to the size of the problem's search space, which, in turn, is related to the choice of τ, UCL_s, etc. Therefore, as the number of components and/or the life cycle length are increased, a dramatic increase is also observed in both the search space and the simulation time. The complexity of the problem also increases with a decrease of τ, as in this case the number of non-periodic inspections increases, and the possible number of inspection policies quickly explodes. Thus, based on all of these, a reduction of computational complexity and an increase in computational efficiency are required.

Reducing the search space and problem complexity can be achieved by decreasing the number of instances of calculating the total expected cost. The genetic algorithm provides a powerful heuristic search means to do this. The proposed approach is further discussed below.

6. Numerical Examples

The present section provides examples for each of the models developed in the preceding sections.

6.1. Model 1: k-out-of-n system with opportunistic inspections

We first consider a 3-out-of-5 redundant system with all components in the "as-good-as-new" state and parameters given in Table 1.

Table 1. Parameters and Costs of the Power Law Intensity Function.

Case	β	η (months)	Minimal repair cost, c^M	Replacement cost, c^R	Component downtime cost, c^D	System downtime cost, c^{SD}	Fixed inspection cost, c^i
1	1.5	3.5	$75	$200	$60	$550	$50
2	1.5	3.5	$75	$200	$80	$550	$50
3	1.5	3.5	$75	$200	$60	$350	$50
4	1.5	3.5	$75	$200	$60	$550	$100
5	1.5	5	$75	$200	$60	$550	$50

All models were implemented in MATLAB, and "ga" function was called for the genetic algorithm calculations. We took 200 as the number of individuals in a population of candidate solutions and 1 as the number of individuals guaranteed to survive to the next generation ("elite count" parameter). We used gene-based coding scheme, where each element in the coded "genome" string could change independently. Each coded string represented an inspection policy, where the string's length corresponded to the number of possible inspections within the planning horizon less one (since an inspection is always done at the end of the planning horizon according to assumption S.1/2.3), which was 11 for the planning horizon of 1 year with a monthly inspection frequency. Binary alphabet was used for coding, with '1' denoting inspection and '0' denoting no inspection, as mentioned previously. Crossover was performed by taking the weighted average of the parents *parent1* and *parent2* according to the formula:

$$child = parent1 + rand * (parent2 - parent1), \quad (9)$$

where *rand* is a random number. The children thus produced are then within the hypercube defined by the parents at the opposite vertices. The crossover fraction, which is the fraction of the next generation (apart from the elite individuals) produced by crossover, was taken at the default value of 0.8. Adaptive feasible mutation was used, which randomly generated directions that were adaptive toward the last generation. A step length of 1 was chosen along each direction, so that linear constraints and bounds were satisfied. Taking into consideration the running time of the simulation and the error in the calculated mean values, the simulation procedure was repeated for 5000 runs for each of the possible inspection policies with the planning horizon of 12 months and τ = 1 month. The values of the calculated total expected costs were saved and then averaged for each inspection policy. Optimisation was then performed using genetic algorithm, and the output of the GA for Case 1 is given in Fig. 3.

For the GA search procedure, the highest fitness value is equivalent to the lowest total expected cost. The genetic algorithm search used a limit of 100 generations, a stall generation limit of 50, an elite count of 1 and the tolerance limit of 10^{-5}. We repeated the GA for 30 trials with different seeds and chose the best solution with the lowest total expected cost.

As can be seen from Fig. 3, the best solution was found after 11 generations. The top of Fig. 3 plots the best and the mean values of the penalty function for each generation. Because the formulation is integer problem with constraints, the penalty function includes a term for infeasibility. If the generation results in a feasible solution, the penalty function is identical to the fitness function, which is the total expected cost for each generation. Otherwise, the penalty function is the maximum fitness function among the feasible generations plus the sum of the constraint violations of the infeasible points. This ensures that an infeasible solution is not selected in the optimisation process.

The fitness function approaches optimality in the sense of the total expected cost with the generations' number increasing, and converges after about 40 generations. The global optimality has been verified for the periodic optimisation of the *k*-out-of-*n* system by comparing the results from both exhaustive search and genetic algorithm search procedure, which have been found to be identical. Moreover, we found the costs to be very close, so that any locally optimal cost is not far from the globally optimal result obtained from the simulation.

The middle of Fig. 3 shows the average nearest neighbour distances for each generation. Generally, lower distance implies a more localised search after 40 generations.

The bottom of Fig. 3 displays the best, worst and mean fitness function scores. One can notice that starting at generation 39, the dis-

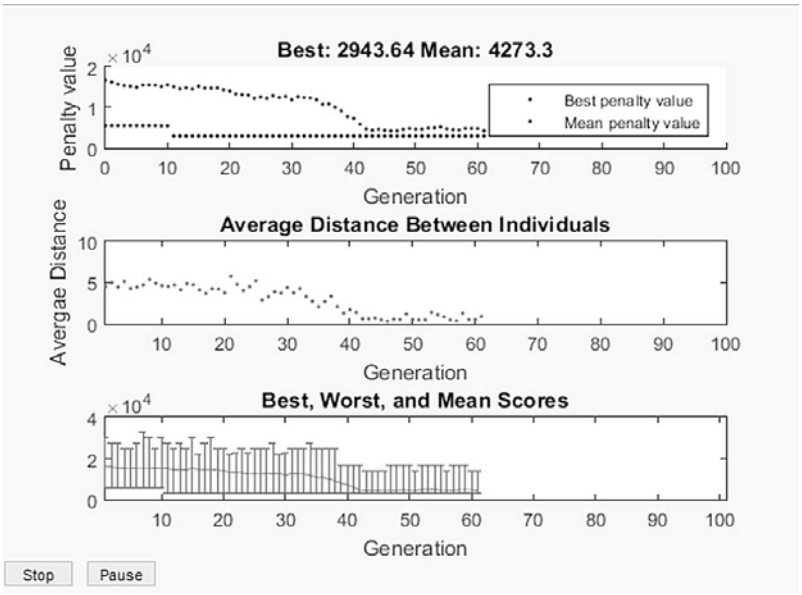

Fig. 3. Genetic algorithm's output and results for Case 1 of a 3-out-of-5 system

tance between the best and the worst scores (i.e. total expected costs) decreases, while the mean score approaches the best score.

Baseline Case 1 was compared with the other cases for sensitivity analysis. The results of the genetic algorithm's search procedure for the cases are given in Table 2. Note that the last column in the table represents the number of distinct inspection policies out of 2048 inspection policies. The fact that optimal policies were found well before reaching the total number of policies suggests fast convergence of the genetic algorithm in the neighbourhood of the optimal cost. Such quick convergence toward the optimal solution is likely resulting from the smoothness of search space in vicinity of the optimal solution.

Table 2 shows that the optimal inspection policy suggests that for the baseline Case 1 system, 9 inspections should be performed at month 1, 3, 4, 6, 7, 8, 9, 11 and 12 within the 12-months planning horizon, and the optimal maintenance policy suggests that failed components should be replaced upon failure after having been minimally repaired for 5 times.

For the system with the component downtime cost of \$80 (Case 2), 9 inspections should be done at month 2, 3, 5, 6, 7, 9, 10, 11 and 12, and the optimal maintenance policy prescribes component replacement on 7th failure within the 12-months planning horizon. Unsurprisingly, the total expected cost for Case 2 is greater than that for Case 1, since the cost of component downtime is greater for Case 2 than for Case 1.

For Case 3, the lower downtime cost for the system results in lower $E\left[C^{T,k,n}_{\boldsymbol{x}^*,m^*}\right]$, fewer inspections and greater m^* compared with Case 1. Lower system downtime penalty translates into more of allowable system downtime, which necessitates fewer inspections (7 for Case 3 vs. 9 for Case 1) and a much greater m. The fact that $m^* = UCL$ implies that it is economically infeasible to replace failed components when the system downtime penalty is significantly decreased. However, the total effect of the decrease of the system downtime penalty by \$200 is reduced by the increased downtime as a result of the fewer inspections and greater component deterioration due to fewer replacements, all of which are reflected in the total expected cost's decrease of only \$168.01.

The optimal joint inspection and maintenance policies for Case 4 are close to those of Case 3, but the total expected cost is greater than that for either Case 3 or Case 1. Unsurprisingly, increasing the cost of system inspection results in the increase in $E\left[C^{T,k,n}_{\boldsymbol{x}^*,m^*}\right]$. Removing the effect of the total cost of inspection, it can be seen that the remaining expected cost for Case 4 is lower than that for Case 1 by \$70.47. This is likely the result of the 2-fold increase in the optimal number of minimal repairs before replacement, which results in fewer component replacements prescribed by Case 4 compared with those for Case 1.

Finally, for Case 5, the total expected cost is the lowest among all the tested cases. This can be explained by higher scale (spread) parameter of the time-to-failure distribution, which implies fewer failures within the same time interval for Case 5 compared to the other cases. Using Equation (11) from Babishin and Taghipour [3], the calculated expected number of system failures is approximately 5.7 for Cases 1-4 and only about 2.6 for Case 5 – a decrease by over 121 % for Case 5 compared to the other cases. This also results in the fewest optimal number of inspections (6) among all the cases and, also, a slightly higher optimal number of minimal repairs until replacement (7) and, correspondingly, fewer component replacements compared to that for Case 1.

6.2. Model 2: system with hard-type and soft-type components and opportunistic inspections

We consider a system composed of m_1=5 components prone to hidden soft failure and m_2=3 components prone to hard failure, all of which are initially "as-good-as-new". Two cases are considered: Case 1 (Baseline) and Case 2 (1.5-time greater monthly downtime penalty cost compared to Baseline).The input parameters for the failure distributions, the costs of minimal repair, replacement and downtime are given for both cases in Table 3.

The fixed cost of scheduled inspections c_I=\$25. Both cases are simulated for 1000 runs with the planning horizon of 12 months and τ = 1 month. Lower number of simulation runs had to be used because of the much greater time required to run the simulation and the genetic algorithm for the hard-and-soft-type system compared to a *k*-out-of-*n* system, which is a result of the former's greater complexity.

The output of the GA for Case 1 is given in Fig. 4. The genetic algorithm search used a limit of 100 generations, a stall generation limit of 50, an elite count of 1 and the tolerance limit of 10^{-5}.

As can be seen from the top graph in Fig. 4, the best solution ("best penalty value") was found after 71 generations.The global optimality has been verified by comparing the results from both exhaustive search and genetic algorithm search procedure, which have been found to be identical.

Table 2. Optimal policies from the genetic algorithm

Case #	Distinction from baseline	Total expected cost, $E\left[C^{T,k,n}_{\boldsymbol{x}^*,m^*}\right]$	Optimal inspection policy, $\boldsymbol{x}^*$	Optimal maintenance policy, m^*	Number of inspection policies analysed, a
1	—	\$2943.64	(1 0 1 1 0 1 1 1 1 0 1 1)	5	1755
2	c^D = \$80	\$3098.46	(0 1 1 0 1 1 1 0 1 1 1 1)	6	1654
3	c^{SD} = \$350	\$2775.63	(1 0 1 0 1 1 0 1 1 0 0 1)	11	1709
4	c^i = \$100	\$3123.17	(1 0 1 1 0 1 0 1 0 1 0 1)	10	1638
5	η = 5	\$1946.68	(0 1 0 0 1 1 0 0 1 1 0 1)	7	1104

Table 3. Power law intensity parameters and costs for different components of Case 1 (Baseline) and Case 2.

Component type		β_j	η_j (months)	Minimal repair cost, c_j^M	Replacement cost, c_j^R	Case 1 downtime penalty cost/ month, c_j^D	Case 2 downtime penalty cost/ month, c_j^D
Soft	1	1.3	3.5	\$70	\$200	\$80	\$120
	2	2.8	4.6	\$45	\$150	\$55	\$82.5
	3	2.1	2.7	\$100	\$300	\$85	\$127.5
	4	3.2	7.0	\$75	\$240	\$90	\$135
	5	1.7	3.6	\$125	\$325	\$100	\$150
Hard	1	1.5	8.7	–	–	–	–
	2	1.2	6.0	–	–	–	–
	3	1.7	7.5	–	–	–	–

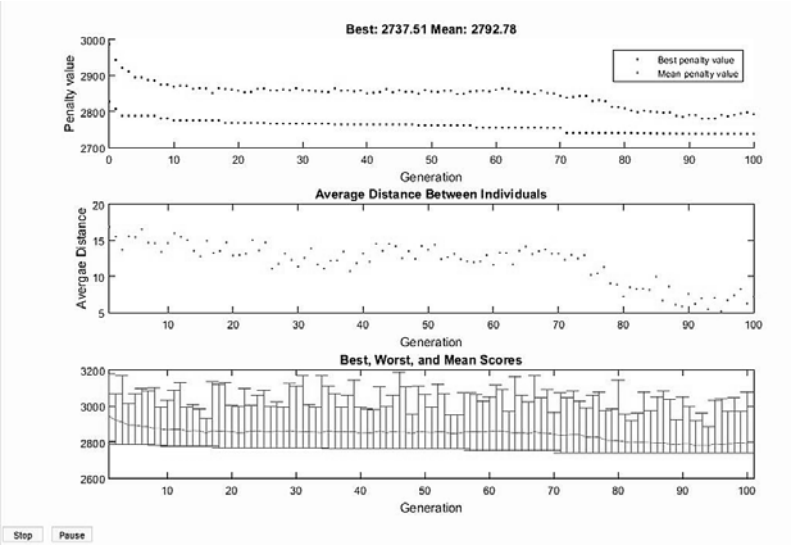

Fig. 4. Genetic algorithm's output and results for Case 1 of a system composed of soft- and hard-type components.

The graph in the middle of Fig. 4 shows the nearest neighbour distances for each population member. Overall, lower distance implies the search space narrowing down starting at around the 80th generation on.

The graph at the bottom in Fig. 4 contains the best, worst and mean fitness function scores. One can see the mean scores of the fitness function are minimal at generations 92-94, owing to the lowest worst scores, while the best scores remain unchanged starting at generation 71.

The soft-type and the hard-type component parameters from Table 3 are used to optimise for the joint inspection and maintenance policies using genetic algorithm. The GA procedure was repeated for 5 trials, and the best results are shown. The results are obtained for both Case 1 and Case 2. The optimal maintenance policies, i.e. the optimal numbers of minimal repair until replacement and the optimal replacement ages, are provided for both cases in Table 4.

As can be seen from Table 4, the optimal replacement ages for some of the hard-type components (namely, for hard-type component 1 for Case 1 and hard-type components 1 and 3 for Case 2) exceed the planning horizon. This simply means that these components would be replaced only at the end of the planning horizon and would be minimally repaired if they fail at any time until then. Also, changes in the optimal maintenance policies for components with soft and hard failures suggest that they are affected by changes in the component downtime penalty.

The resultant optimal inspection policy was also found for both cases to be as shown in Table 5. As the last column of Table 5 suggests, 9935 out of 10240 inspection policies were checked in order to find the optimal one. The timing results suggest that, generally, in the case of non-periodic inspections, validating GA procedure by checking all possibilities is not practical. The advantage of using GA is further supported by Hajipour and Taghipour, who previously found that a genetic algorithm required only about 7 % of the time needed for exhaustive search [16].

Table 4. Optimal maintenance policies for Case 1 and Case 2

Case #	Distinction from baseline	Soft-type component, s	Optimal number of minimal repairs before replacement, m_s^*	Hard-type component, h	Optimal replacement age, ζ_h (months)
1	––	1	7	1	14.46
		2	1	2	8.25
		3	6	3	11.83
		4	8	–	–
		5	13	–	–
2	c^D(Case 2) = 1.5 cD (Case 1)	1	9	1	16.61
		2	1	2	6.77
		3	3	3	13.98
		4	0	–	–
		5	10	–	–

The optimal inspection policy for Case 1 implies that the system is inspected 4 times in months 5, 8, 10 and 12. The optimal inspection policy for Case 2 is drastically different with 10 inspections occurring on a monthly basis in months 3–12. Thus, as a result of a 1.5-time increase in the per-component monthly downtime penalty cost, the optimal inspection policy alone for Case 2 costs \$150 more (\$25 · (10−4) inspections) than that for Case 1. This leaves another \$960.90 as the increase in the cost of the optimal maintenance policy out of the total increase of \$1110.90 (\$3848.41-\$2737.51) in the total expected cost for Case 2 compared to Case 1. Converting dollars into percent-

Table 5. Optimal policies from the genetic algorithm

Case #	Total expected cost, $E\left[C^{T,HS}_{\mathbf{x}^a,m_s,\zeta_h}\right]$	Optimal inspection policy, $\mathbf{x}^*$	Number of inspection policies analysed, a
1	\$2737.51	(0 0 0 0 1 0 0 1 0 1 0 1)	9935
2	\$3848.41	(0 0 1 1 1 1 1 1 1 1 1 1)	9935

Table 6. Absolute and relative changes in the expected costs for Case 2 compared to Case 1.

Change (Case 2 – Case 1)	Average system downtime penalty cost	Average optimal replacement ages	Average optimal number of minimal repairs before replacement	Optimal inspection policy cost	Optimal maintenance policy cost	Total expected cost, $E\left[C^{T,HS}_{\boldsymbol{x}^a,m_s,\zeta_h}\right]$
Absolute	+ \$205	+ 0.94	- 2.40	+ \$150	+ \$960.90	+ \$1110.90
Relative	+ 50 %	+ 8.13 %	- 52.17 %	+ 5.48 %	+ 35.10 %	+ 40.58 %

ages, the total expected cost is over 40 % greater for Case 2 compared for Case 1. A large reduction in m_s^* for 3 of 5 soft-type components is meant to decrease these components' downtime by reducing their ages to 0 with each replacement more frequently for Case 2 than for Case 1. Averaged across all soft-type components, m_s^* =4.6 for Case 2 and m_s^* =7 for Case 1 – a decrease of over 52 % as a result of the downtime costs increasing by 50 %. At the same time, the optimal replacement ages of the hard-type components increased, on average, by only slightly over 8 % (which is less than τ) for Case 2 compared to Case 1. These results are summarised inTable 6.

As can be concluded from Table 6, the average optimal number of minimal repairs before replacement is most sensitive to change in the downtime penalty, followed by the total expected cost and the optimal maintenance policy cost, with an increase in downtime penalty causing a decrease in m_s^* and increase in each of $E\left[C^{T,HS}_{\boldsymbol{x}^a,m_s,\zeta_h}\right]$ and the optimal maintenance policy cost. On the contrary, the optimal inspection policy cost is the least sensitive, followed by the average optimal replacement ages, where an increase in downtime penalty increases each of the optimal inspection policy cost and the average optimal replacement ages.

7. Conclusions

In the present article, optimisation of non-periodic maintenance and inspection was considered for complex multicomponent systems with either *k*-out-of-*n* redundant configuration, or with components prone to hard and soft failures. Aside from scheduled inspections, components can be checked opportunistically at system failures. Making the unit of time sufficiently small allows to treat the planning horizon as quasi-continuous for possible non-periodic inspections, which gives a much greater flexibility and variety in the choice of available inspection policies at the expense of computational complexity, when compared to the periodic inspections. Since soft failures are hidden, component's age cannot be used as the criterion for maintenance optimisation. Instead, maintenance policies are defined by the number of minimal repairs before replacement for each component prone to hidden soft failure. The optimal policies are then found by jointly optimising the inspection and maintenance policies for the lowest total expected cost. Using simulation and a genetic algorithm to implement the joint optimisation was found to be an efficient and convenient method to find the optimal policies for large and complex systems. This appears to be a promising method for optimisation with regards to complex systems with multiple decision parameters.

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Appendix

This section presents the recursive functions for calculating the expected values in the simulation. The general principle of constructing a generator function is to calculate the expectation of an event of interest, such as a failure, minimal repair, replacement, or uptime. Since components are not independent of each other, and their survival is related to survival of other components, it is necessary to condition their probabilities on reliabilities and failures of other components. For example, replacement of a soft-type component is conditional on its (UCL_s+1)st failure and inspection, which itself is conditional on the failure of any hard-type component and inspection policy. In order to make the generator function general and applicable to different categories of components and events, placeholder functions are used extensively. Depending on the placeholder function $\psi(y,z,I_s)$, either $M_{\boldsymbol{x}^a,m_s,s}(\omega,t_s,\boldsymbol{\theta},f_s,\boldsymbol{\zeta},m_c)$, $R_{\boldsymbol{x}^a,m_s,s}(\omega,t_s,\boldsymbol{\theta},f_s,\boldsymbol{\zeta},m_c)$, or $U_{\boldsymbol{x}^a,m_s,s}(\omega,t_s,\boldsymbol{\theta},f_s,\boldsymbol{\zeta},m_c)$ functions are obtained. For $\psi(y,z,I_s)$, I_s is 0, 1, or 2, if component s is, respectively, minimally repaired, replaced, or has not failed in an interval. The first failure times of soft-type component s and the hard subsystem are, respectively, random variables Y and Z, and their density and reliability functions are $f^Y(y\,|\,t_s)$, $f^Z(y|\boldsymbol{\theta}),\mathcal{R}^Y(y|t_s)$ and $\mathcal{R}^Z(z|\boldsymbol{\theta})$. We then formulate the generator function for the expected uptime and the expected numbers of replacements and minimal repairs for each component over inspection interval $[0,i\tau]$ by conditioning on the random variables $Y=y$ and $Z=z$. Taking $q^{k,n}(z)$ to be the probability that the failure of the system at time $Z=z$ is due to the failure of component s:

$$q^{k,n}(z)f^Z(z|t_s)=\lambda_s(z\,|\,t_s)\exp\left(-\int_0^z\lambda^H(\chi\,|\,t_s)d\chi\right).$$

We start by assuming only one scheduled inspection at the end of the planning horizon and work backward to initial time 0. For the system with soft-type and hard-type components, the expected number of minimal repairs for a soft-type component can be calculated as following:

$$G_{x_1,m_s,s}(\omega,t_s,\boldsymbol{\theta},f_s,\boldsymbol{\zeta},m_c)=\int_0^{i\tau}\sum_{0}^{z}\sum_{h=1}^{n_2}\Big\{\Big(\Big[\Big[\psi(y,z,0)+G_{x_1-z,m_s,s}\left(\omega-z,t_s+y,\boldsymbol{\theta}\oplus z,f_s+1,\boldsymbol{\zeta}^*,m_c+1\right)\Big]r_s^Y(m_c,m_s)$$
$$+\Big[\psi(y,z,1)+G_{x_1-z,m_s,s}\left(\omega-z,0,\boldsymbol{\theta}\oplus z,0,\boldsymbol{\zeta}^*,0\right)\Big]\bar{r}_s^Y(m_c,m_s)\Big)r^{k,\mathbf{n}}(f_s,k)$$
$$+\Big(\Big[\psi(y,z,0)+G_{x_1-z,m_s,s}\left(\omega-z,t_s+y,(\boldsymbol{\theta}\oplus z)^{(0_h)},f_s+1,\boldsymbol{\zeta}^*,m_c+1\right)\Big]r_s^Y(m_c,m_s)$$
$$+\Big[\psi(y,z,1)+G_{x_1-z,m_s,s}\left(\omega-z,0,(\boldsymbol{\theta}\oplus z)^{(0_h)},0,\boldsymbol{\zeta}^*,0\right)\Big]\bar{r}_s^Y(m_c,m_s)\Big)\bar{r}^{k,n}(f_s,k)\Big\}q^h(z)f^Y(y\,|\,t_s)f^Z(z|t_s)\,dydz$$
$$+\int_0^{i\tau}\sum_{s=1}^{n_1}\Big\{\Big[\psi(z,z,2)+G_{x_1-z,m_s,s}\left(\omega-z,t_s+z,\boldsymbol{\theta}\oplus z,f_s,\boldsymbol{\zeta}^*,m_c\right)\Big]r^{k,n}(f_s,k)$$
$$+\Big[\psi(z,z,2)+G_{x_1-z,m_s,s}\left(\omega-z,t_s+z,(\boldsymbol{\theta}\oplus z)^{(0_h)},f_s,\boldsymbol{\zeta}^*,m_c\right)\Big]\bar{r}^{k,\mathbf{n}}(f_s,k)\Big\}q^{k,n}(z)\mathcal{R}^Y(z\,|\,t_s)f^Z(z|t_s)\,dz$$
$$+\int_0^{i\tau}\Big[\psi(y,i\tau,0)r_s^Y(m_c,m_s)+\psi(y,i\tau,1)\bar{r}_j^Y(m_c,m_s)\Big]f^Y(y\,|\,t_j)dy\mathcal{R}^Z(i\tau|\boldsymbol{\theta})$$
$$+\psi(i\tau,i\tau,2)\mathcal{R}^Y(i\tau\,|\,t_j)\mathcal{R}^Z(i\tau|\boldsymbol{\theta}),\tag{A.1}$$

where:

$$r_s^Y(m_c,m_s)=\begin{cases}1, & if\ m_c\le m_s\\ 0, & otherwise\end{cases},$$

$$r^{k,n}(f_s,k)=\begin{cases}1, if\ f_s<n-k\\ 0, otherwise\end{cases},$$

$$\bar{r}_s^Y=1-r_s^Y,\ \bar{r}_h^Z=1-r_h^Z.$$

The placeholder function $\psi(y,z,I_s)$ varies depending on the random variable of interest.

For the expected number of minimal repairs:

$$\psi(y,z,I_s)=\begin{cases}1, if\ I_s=0\\ 0, otherwise\end{cases}.$$

For the expected number of replacements:

$$\psi(y,z,I_s)=\begin{cases}1, if\ I_s=1\\ 0, otherwise\end{cases}.$$

For the expected uptime:

$$\psi(y,z,I_s)=\begin{cases}y, if\ \ y<z\\ z, otherwise\end{cases}.$$

These placeholder functions modify the generator function $G_{x_1,m_s,s}(\omega,t_s,\boldsymbol{\theta},f_s,\boldsymbol{\zeta},m_c)$, which is used for the first inspection (initial state), and then, through recursion, in $G_{x_i,m_s,s}(\omega,t_s,\boldsymbol{\theta},f_s,\boldsymbol{\zeta},m_c)$ (see below and Equation (A.2)). Thus, the values of uptime and number of minimal repairs before replacement are obtained from Monte Carlo simulation. For a particular component in one run, its expected uptime is obtained from simulating this component in conjunction with the other components for the length of the entire planning horizon, recording its downtime, and then subtracting its downtime from the planning horizon to obtain its uptime. This can be repeated and averaged to obtain the component's expected uptime.

For the number of minimal repairs before replacement, the upper confidence limit on the expected number of component failures is obtained from Equations 1.1-2.1, or 1.2-2.2 for each soft-type component. Simulations are then run for all values of m or m_s, starting from 0 and up to and including UCL or UCL_s.

$\boldsymbol{\theta} \oplus z$ denotes addition of a scalar z to each of the coordinates of vector $\boldsymbol{\theta}$, i.e. $\boldsymbol{\theta} \oplus z = \left(\theta_1 + z, \theta_2 + z, \ldots, \theta_{n_2} + z\right)$, and $\left(\boldsymbol{\theta} \oplus z\right)^{(0_h)}$ means the h^{th} coordinate of vector $\boldsymbol{\theta} \oplus z$ is replaced by zero, i.e. $\left(\boldsymbol{\theta} \oplus z\right)^{(0_h)} = \left(\theta_1 + z, \ldots, \theta_{h-1} + z, 0, \theta_{h+1} + z, \ldots, \theta_{n_2} + z\right)$.

Equation (A.1) is extended to obtain the expected value of a random variable of interest for inspection policy x_i when inspections are performed at $i\tau$:

$$
\begin{aligned}
G_{x_i,m_s,s}(\omega,t_s,\boldsymbol{\theta},f_s,\zeta,m_c) = \int_0^{i\tau}\sum_{h=1}^{n_2}\Big\{\Big[\Big[\psi(y,z,0)+G_{x_i-z,m_s,s}\left(\omega-z,t_s+y,\boldsymbol{\theta}\oplus z,f_s+1,\zeta^*,m_c+1\right)\Big]r_s^Y(m_c,m_s)\\
+\Big[\psi(y,z,1)+G_{x_i-z,m_s,s}\left(\omega-z,0,\boldsymbol{\theta}\oplus z,0,\zeta^*,0\right)\Big]\bar{r}_s^Y(m_c,m_s)\Big)r^{k,\mathbf{n}}(f_s,k)\\
+\Big(\Big[\psi(y,z,0)+G_{x_i-z,m_s,s}\left(\omega-z,t_s+y,(\boldsymbol{\theta}\oplus z)^{(0_h)},f_s+1,\zeta^*,m_c+1\right)\Big]r_s^Y(m_c,m_s)\\
+\Big[\psi(y,z,1)+G_{x_i-z,m_s,s}\left(\omega-z,0,(\boldsymbol{\theta}\oplus z)^{(0_h)},0,\zeta^*,0\right)\Big]\bar{r}_s^Y(m_c,m_s)\Big)\bar{r}^{k,n}(f_s,k)\Big\}q^{k,n}(z)f^Y(y\,|\,t_s)r^{k,n}(f_s,k)\,dydz\\
+\int_0^{i\tau}\sum_{s=1}^{n_1}\Big\{\Big[\psi(z,z,2)+G_{x_i-z,m_s,s}\left(\omega-z,t_s+z,\boldsymbol{\theta}\oplus z,f_s,\zeta^*,m_c\right)\Big]r^{k,n}(f_s,k)\\
+\Big[\psi(z,z,2)+G_{x_i-z,m_s,s}\left(\omega-z,t_s+z,(\boldsymbol{\theta}\oplus z)^{(0_h)},f_s,\zeta^*,m_c\right)\Big]\bar{r}^{k,\mathbf{n}}(f_s,k)\Big\}q^{k,n}(z)\mathcal{R}^Y(z\,|\,t_s)f^Z(z|t_s)\,dz\\
+\int_0^{i\tau}\Big[\Big(\psi(y,i\tau,0)+G_{x_{i-1},m_s,s}\left(i\tau,t_s+y,\boldsymbol{\theta}\oplus i\tau,f_s+1,\zeta^*,m_c+1\right)\Big)r_s^Y(m_c,m_s)+\Big(\psi(y,i\tau,1)+G_{x_{i-1},m_s,s}\left(i\tau,0,\boldsymbol{\theta}\oplus i\tau,0,\zeta^*,0\right)\Big)\bar{r}_j^Y(m_c,m_s)\Big]f^Y(y\,|\,t_s)dy\mathcal{R}^Z(\sigma|t_s)\\
+\Big(\psi(i\tau,i\tau,2)+G_{x_{i-1},m_s,s}\left(i\tau,t_s+i\tau,\boldsymbol{\theta}\oplus i\tau,f_s,\zeta^*,m_c\right)\Big)\mathcal{R}^Y(i\tau|t_s)\mathcal{R}^Z(i\tau|t_s).
\end{aligned}
\tag{A.2}
$$

Equation (A.2) is provided for a given inspection policy index a and has to be applied for all inspection policies. To get an idea about how the generator function is constructed, consider the first line in Equation (A.2). The placeholder function $\psi(y,z,0)$ and the recursive term $G_{x_i-z,m_s,s}\left(\omega-z,t_s+y,\boldsymbol{\theta}\oplus z,f_s+1,\zeta^*,m_c+1\right)$ contribute to the expected number of minimal repairs, if $r_s^Y(m_c,m_s)=1$, i.e. if the number of minimal repairs before replacement is lower than the optimal value. The other terms are responsible for the expected number of repairs, or the expected uptime.

The hazard rate of a hard subsystem consisting of all hard-type components in series configuration is given as $\lambda^H(z\,|\,\boldsymbol{\theta}) = \sum_{h=1}^{n_2}\lambda_h(\theta_h+z)$.

For the system with hard-type and soft-type components, Equations (A.1-A.2) can still be used, if function $r^{k,n}(f_s,k)$ is redefined as $r_h^Z\left(z,\zeta_h^*\right)$:

$$
r_h^Z\left(z,\zeta_h^*\right) = \begin{cases} 1, \textit{ if } z \le \zeta_h^* \\ 0, \textit{ otherwise} \end{cases},
$$

and $q^{k,n}(z)$ is redefined as $q^h(z)$, which is the probability that the failure of the hard subsystem at time $Z=z$ is due to the failure of hard-type component h:

$$
q^h(z)f^Z(z|\boldsymbol{\theta}) = \lambda_h(z\,|\,\boldsymbol{\theta}_h)\exp\left(-\int_0^Z\lambda^H(\chi\,|\,\theta)d\chi\right).
$$

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