10.2478/v10170-011-0021-3

Archives of Control Sciences
Volume 22(LVIII), 2012
No. 2*, pages 217–231*

LMI based control design for linear systems with distributed time delays

ANNA FILASOVÁ, DANIEL GONTKOVIČ and DUŠAN KROKAVEC

The paper concerns the problem of stabilization of continuous-time linear systems with distributed time delays. Using extended form of the Lyapunov-Krasovskii functional candidate, the controller design conditions are derived and formulated with respect to the incidence of structured matrix variables in the linear matrix inequality formulation. The result give sufficient condition for stabilization of the system with distributed time delays. It is illustrated with a numerical example to note reduced conservatism in the system structure.

Key words: linear matrix inequality, systems with distributed time delays, Lyapunov-Krasovskii functional, state control, asymptotic stability

1. Introduction

Control systems are used in many industrial applications, where time delays can take a deleterious effect on stability and dynamic performance in open and closed-loop systems. Therefore the stability and control of the dynamical systems involving distributed time delays is the problem of a great interest and intensive activities are done to develop control laws for systems stabilization.

The use of Lyapunov method for stability analysis of the time delay systems has been a growing subject of interest, starting with the pioneering works of Krasovskii [10], [11]. Currently, for the stability issue, modified Lyapunov-Krasovskii functionals are used to obtain delay-independent stabilization and the results based on these functionals are applied to controller synthesis and observer design. This time-delay independent methodology, and the bounded inequality techniques are sources of a conservatism that can cause higher norm of the state feedback gain. Much research has been done and stability criteria have been derived for systems with time-delays in state variables (e.g.

The authors are with Technical University of Košice, Faculty of Electrical Engineering and Informatics, Department of Cybernetics and Artificial Intelligence, Letná 9, 042 00 Košice, Slovakia, fax: +421 55 602 2564. E-mails: anna.filasova@tuke.sk, daniel.gontkovic@tuke.sk, dusan.krokavec@tuke.sk

The work presented in this paper was supported by Grant Agency of Ministry of Education and Academy of Science of Slovak Republic VEGA under Grant No. 1/0256/11. This support is very gratefully acknowledged.

Received 5.05.2012.

in [14], [16]), especially formulated with respect to LMI principles ([1], [5], [9]). Some review of progress in this research field can be found e.g. in [19], [21].

Systems with distributed time delays were applied e.g. in the modeling of combustion chambers rocket motor with pressure feeding [4], [20]. Because of the significance of systems with distributed time-delay, as the controllers are usually digitally implemented, a growing attention has been devoted to studying distributed delay systems in recent years. Reflecting the fact that standard time-delay control design schemes are not applicable to systems with distributed time delays, new stability conditions had to be derived (e.g. in [8], [17]). The readers are referred to [3], and the reference therein, for recent reports about the stability analysis of these systems.

The presented approach bases on the extended form of Lyapunov-Krasovskii functional, established by introducing triple integral terms [18], where the stability conditions as well as the controller design method are reformulated with respect to the application of structured matrix variables, and by employing the integral partitioning technique [7]. Ideas in this direction can be found in [2], as well as in the authors preliminary results published in [5], [13]. Generally, since Lyapunov-Krasovskii functional is used, only sufficient conditions for system stability are obtained.

The outline of this paper is as follows: section 2 briefly introduces the model of continuous-time linear MIMO systems with distributed time delays, and in section 3 the basis preliminaries are derived. According to the system model properties, new LMI structures, specifically designed to respect the structural LMI variables implementation in LMI solvers, are introduced in section 4 in accordance to stability conditions of the autonomous system. These conditions form the basis on which the control law parameter design conditions are derived in section 5. Finally, in section 6 a numerical example is presented to illustrate basic properties of the presented method. Section 7 presents concluding remarks.

Throughout the paper, the following notations are used: $\boldsymbol{x}^T$, $\boldsymbol{X}^T$ denote the transpose of the vector $\boldsymbol{x}$ and matrix $\boldsymbol{X}$, respectively, $diag[\cdot]$ denotes a block diagonal matrix; a square matrix $\boldsymbol{X} > 0$ (respectively $\boldsymbol{X} < 0$) means that $\boldsymbol{X}$ is a symmetric positive definite matrix (respectively, negative definite matrix); the symbol $\boldsymbol{I}_n$ represents the n-th order unit matrix, $\mathbb{R}$ denotes the set of real numbers and $\mathbb{R}^{n\times r}$ the set of all $n \times r$ real matrices.

2. System model

The systems under consideration are understood as multi-input and multi-output linear (MIMO) dynamic systems with distributed time delay. Without lose of generalization, this class of systems can be represented in the state-space form by the set of equations

$$\dot{\boldsymbol{q}}(t) = \boldsymbol{A}\boldsymbol{q}(t) + \boldsymbol{A}_h \int_{t-h}^{t} \boldsymbol{q}(s)\mathrm{d}s + \boldsymbol{B}\boldsymbol{u}(s) \tag{1}$$

$$\boldsymbol{y}(t) = \boldsymbol{C}\boldsymbol{q}(t) + \boldsymbol{D}\boldsymbol{u}(t) \tag{2}$$

with initial conditions

$$\boldsymbol{q}(\theta) = \varphi(\theta), \quad \forall\theta \in \langle -(h+\frac{h}{m}), 0\rangle \tag{3}$$

where $h > 0$ represents the delay, $m > 0$ is a partitioning factor, $\boldsymbol{q}(t) \in \mathbb{R}^n$, $\boldsymbol{u}(t) \in \mathbb{R}^r$, and $\boldsymbol{y}(t) \in \mathbb{R}^p$ are vectors of the state, input and output variables, respectively, and matrices $\boldsymbol{A} \in \mathbb{R}^{n\times n}$, $\boldsymbol{A}_h \in \mathbb{R}^{n\times n}$, $\boldsymbol{B} \in \mathbb{R}^{n\times r}$, $\boldsymbol{C} \in \mathbb{R}^{p\times n}$, and $\boldsymbol{D} \in \mathbb{R}^{p\times r}$ are real matrices.

Using the linear memoryless state feedback controller

$$\boldsymbol{u}(t) = -\boldsymbol{K}\boldsymbol{q}(t) \tag{4}$$

where the matrix $\boldsymbol{K} \in \mathbb{R}^{r\times n}$ is the gain matrix, the problem of interest is to design $\boldsymbol{K}$ such that the closed-loop system

$$\dot{\boldsymbol{q}}(t) = (\boldsymbol{A} - \boldsymbol{B}\boldsymbol{K})\boldsymbol{q}(t) + \boldsymbol{A}_h \int_{t-h}^{t} \boldsymbol{q}(s)\mathrm{d}s \tag{5}$$

is asymptotically stable for given h.

3. Basic preliminaries

Assumption 1 The couple $(\boldsymbol{A}, \boldsymbol{B})$ is controllable.

Proposition 1 *If $\boldsymbol{N}$ is a positive definite symmetric matrix, and $\boldsymbol{M}$ is a square matrix of the same dimension then*

$$\boldsymbol{M}^{-T}\boldsymbol{N}\boldsymbol{M}^{-1} \geqslant \boldsymbol{M}^{-1} + \boldsymbol{M}^{-T} - \boldsymbol{N}^{-1} \tag{6}$$

Proof Since $\boldsymbol{N}$ is positive definite then it yields

$$(\boldsymbol{M}^{-1} - \boldsymbol{N}^{-1})^T \boldsymbol{N} (\boldsymbol{M}^{-1} - \boldsymbol{N}^{-1}) \geqslant 0 \tag{7}$$

$$\boldsymbol{M}^{-T}\boldsymbol{N}\boldsymbol{M}^{-1} - \boldsymbol{M}^{-T} - \boldsymbol{M}^{-1} + \boldsymbol{N}^{-1} \geqslant 0 \tag{8}$$

respectively, and evidently (8) implies (6). This concludes the proof. □

Proposition 2 *(Schur Complement) Let $\boldsymbol{S}$, $\boldsymbol{Q} = \boldsymbol{Q}^T$, $\boldsymbol{R} = \boldsymbol{R}^T$, $\det\boldsymbol{R} \neq 0$ are real matrices of appropriate dimensions, then the next inequalities are equivalent*

$$\begin{bmatrix} \boldsymbol{Q} & \boldsymbol{S} \\ \boldsymbol{S}^T & \boldsymbol{R} \end{bmatrix} > 0 \Leftrightarrow \begin{bmatrix} \mathbf{Q} - \mathbf{S}\mathbf{R}^{-1}\mathbf{S}^T & \mathbf{0} \\ \mathbf{0} & \mathbf{R} \end{bmatrix} > 0 \Leftrightarrow \mathbf{Q} - \mathbf{S}\mathbf{R}^{-1}\mathbf{S}^T > 0,\ \mathbf{R} > 0 \tag{9}$$

Proof (see e.g. [12]) Let the linear matrix inequality takes form

$$\begin{bmatrix} \mathbf{Q} & \mathbf{S} \\ \mathbf{S}^T & \mathbf{R} \end{bmatrix} > 0 \tag{10}$$

then using Gauss elimination principle it yields

$$\begin{bmatrix} \mathbf{I} & -\mathbf{S}\mathbf{R}^{-1} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{Q} & \mathbf{S} \\ \mathbf{S}^T & \mathbf{R} \end{bmatrix} \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ -\mathbf{R}^{-1}\mathbf{S}^T & \mathbf{I} \end{bmatrix} = \begin{bmatrix} \mathbf{Q}-\mathbf{S}\mathbf{R}^{-1}\mathbf{S}^T & \mathbf{0} \\ \mathbf{0} & \mathbf{R} \end{bmatrix} \tag{11}$$

Since

$$\det \begin{bmatrix} \mathbf{I} & -\mathbf{S}\mathbf{R}^{-1} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} = 1 \tag{12}$$

it is evident that this transform doesn't change positivity of (10), and so (11) implies (9). This concludes the proof. □

Proposition 3 *(Symmetric upper-bounds inequalities) Let* $f(\boldsymbol{x}(p))$, $\boldsymbol{x}(p) \in \mathbb{R}^n$, $\boldsymbol{X} = \boldsymbol{X}^T > 0$, $\boldsymbol{X} \in \mathbb{R}^{n\times n}$ *is a real positive definite and integrable vector function of the form*

$$f(\boldsymbol{x}(p)) = \boldsymbol{x}^T(p)\boldsymbol{X}\boldsymbol{x}(p) \tag{13}$$

such that there exist well defined integrations as following

$$\int_{-c}^{0}\int_{t+r}^{t} f(\boldsymbol{x}(p))dp\,dr > 0 \tag{14}$$

$$\int_{t-c}^{t} f(\boldsymbol{x}(p))dp > 0 \tag{15}$$

with $c > 0$, $c \in \mathbb{R}$, $t \in \langle 0, \infty)$, *then*

$$\int_{-c}^{0}\int_{t+r}^{t} \boldsymbol{x}^T(p)\boldsymbol{X}\boldsymbol{x}(p)dp\,dr \geqslant \frac{2}{c^2}\int_{-c}^{0}\int_{t+r}^{t} \boldsymbol{x}^T(p)dp\,dr\,\boldsymbol{X}\int_{-c}^{0}\int_{t+r}^{t} \boldsymbol{x}(p)dp\,dr \tag{16}$$

$$\int_{t-c}^{t} \boldsymbol{x}^T(p)\boldsymbol{X}\boldsymbol{x}(p)dp \geqslant \frac{1}{c}\int_{t-c}^{t} \boldsymbol{x}^T(p)dp\,\boldsymbol{X}\int_{t-c}^{t} \boldsymbol{x}(p)dp \tag{17}$$

Proof (compare e.g. [8], [13]) Using (13) it can be written

$$\boldsymbol{x}^T(p)\boldsymbol{X}\boldsymbol{x}(p)-\boldsymbol{x}^T(p)\boldsymbol{X}\boldsymbol{x}(p)=0 \tag{18}$$

and according to Schur complement (9) it is true that

$$\begin{bmatrix} \boldsymbol{x}^T(p)\boldsymbol{X}\boldsymbol{x}(p) & \boldsymbol{x}^T(p) \\ \boldsymbol{x}(p) & \boldsymbol{X}^{-1} \end{bmatrix} = 0 \tag{19}$$

then the double integration of (19) leads to

$$\begin{bmatrix} \int_{-c}^{0}\int_{t+r}^{t} \boldsymbol{x}^T(p)\boldsymbol{X}\boldsymbol{x}(p)\mathrm{d}p\,\mathrm{d}r & \int_{-c}^{0}\int_{t+r}^{t} \boldsymbol{x}^T(p)\mathrm{d}p\,\mathrm{d}r \\ \int_{-c}^{0}\int_{t+r}^{t} \boldsymbol{x}(p)\mathrm{d}p\mathrm{d}r & \int_{-c}^{0}\int_{t+r}^{t} \boldsymbol{X}^{-1}\mathrm{d}p\,\mathrm{d}r \end{bmatrix} \geqslant 0 \tag{20}$$

Using the equalities

$$\int_{t+r}^{t} \boldsymbol{X}^{-1}\mathrm{d}p = -r\boldsymbol{X}^{-1}, \qquad \int_{-c}^{0} -r\boldsymbol{X}^{-1}\mathrm{d}r = \frac{c^2}{2}\boldsymbol{X}^{-1} \tag{21}$$

inequality (20) can be rewritten as

$$\begin{bmatrix} \int_{-c}^{0}\int_{t+r}^{t} \boldsymbol{x}^T(p)\boldsymbol{X}\boldsymbol{x}(p)\mathrm{d}p\,\mathrm{d}r & \int_{-c}^{0}\int_{t+r}^{t} \boldsymbol{x}^T(p)\mathrm{d}p\,\mathrm{d}r \\ \int_{-c}^{0}\int_{t+r}^{t} \boldsymbol{x}(p)\mathrm{d}p\mathrm{d}r & \frac{c^2}{2}\boldsymbol{X}^{-1} \end{bmatrix} \geqslant 0 \tag{22}$$

It is evident, that (22) implies (16).

Analogously using (19) it yields

$$\begin{bmatrix} \int_{t-c}^{t} \boldsymbol{x}^T(p)\boldsymbol{X}\boldsymbol{x}(p)\mathrm{d}p & \int_{t-c}^{t} \boldsymbol{x}^T(p)\mathrm{d}p \\ \int_{t-c}^{t} \boldsymbol{x}(p)\mathrm{d}p & \int_{t-c}^{t} \boldsymbol{X}^{-1}\mathrm{d}p \end{bmatrix} \geqslant 0 \tag{23}$$

and since

$$\int_{t-c}^{t} \boldsymbol{X}^{-1}\mathrm{d}p = c\boldsymbol{X}^{-1} \tag{24}$$

the following is obtained

$$\begin{bmatrix} \int_{t-c}^{t} \boldsymbol{x}^T(p)\boldsymbol{X}\boldsymbol{x}(p)\mathrm{d}p & \int_{t-c}^{t} \boldsymbol{x}^T(p)\mathrm{d}p \\ \int_{t-c}^{t} \boldsymbol{x}(p)\mathrm{d}p & c\boldsymbol{X}^{-1} \end{bmatrix} \geqslant 0 \tag{25}$$

which implies (17). This concludes the proof. □

4. Stability of the autonomous system

In this section a delay-dependent criterion is presented for asymptotic stability of autonomous (unforced) linear systems with distributed time delays. The formulation bases on the Lyapunov method and LMI approach with structured matrix variables. Note, that the structure of a structured matrix variable can by specified only with matrix variables multiplied by natural numbers.

Theorem 4 *The autonomous system of* (1) *is asymptotically stable if for given* $h > 0$, $m > 0$ *there exist symmetric positive definite matrices* $\boldsymbol{P}, \boldsymbol{U}, \boldsymbol{V} \in \mathbb{R}^{n\times n}$, $\boldsymbol{W} \in \mathbb{R}^{mn\times mn}$ *such that*

$$\boldsymbol{P} = \boldsymbol{P}^T > 0, \quad \boldsymbol{U} = \boldsymbol{U}^T > 0, \quad \boldsymbol{V} = \boldsymbol{V}^T > 0, \quad \boldsymbol{W} = \boldsymbol{W}^T > 0 \tag{26}$$

$$\boldsymbol{P}^\circ = \boldsymbol{T}_A^T \boldsymbol{P} \boldsymbol{T}_I + \boldsymbol{T}_I^T \boldsymbol{P} \boldsymbol{T}_A + \boldsymbol{T}_U^T \boldsymbol{U}^\circ \boldsymbol{T}_U + \boldsymbol{T}_V^T \boldsymbol{V}^\circ \boldsymbol{T}_V + \boldsymbol{T}_W^T \boldsymbol{W}^\circ \boldsymbol{T}_W < 0 \tag{27}$$

where

$$\boldsymbol{T}_U = \begin{bmatrix} a\boldsymbol{I}_n & \\ & a^{-1}\boldsymbol{I}_n \end{bmatrix} \begin{bmatrix} \boldsymbol{I}_n & \boldsymbol{0} & \begin{bmatrix} \boldsymbol{0} & \cdots & \boldsymbol{0} \end{bmatrix} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{I}_n & \begin{bmatrix} \boldsymbol{0} & \cdots & \boldsymbol{0} \end{bmatrix} & \boldsymbol{0} \end{bmatrix} \tag{28}$$

$$\boldsymbol{T}_V = \begin{bmatrix} b\boldsymbol{I}_n & \\ & b^{-1}\boldsymbol{I}_n \end{bmatrix} \begin{bmatrix} \boldsymbol{A} & \boldsymbol{A}_h & \begin{bmatrix} \boldsymbol{A}_h & \cdots & \boldsymbol{A}_h \end{bmatrix} & \boldsymbol{0} \\ a^2\boldsymbol{I}_n & -\boldsymbol{I}_n & \begin{bmatrix} \boldsymbol{0} & \cdots & \boldsymbol{0} \end{bmatrix} & \boldsymbol{0} \end{bmatrix} \tag{29}$$

$$\boldsymbol{T}_W = \begin{bmatrix} \boldsymbol{0}_w & \begin{bmatrix} \boldsymbol{I}_{mn} & \boldsymbol{0}_w \end{bmatrix} \\ \boldsymbol{0}_w & \begin{bmatrix} \boldsymbol{0}_w & \boldsymbol{I}_{mn} \end{bmatrix} \end{bmatrix} \tag{30}$$

$$a = \sqrt{\frac{h}{m}}, \qquad b = \frac{h}{\sqrt{2}m} \tag{31}$$

$$\boldsymbol{U}^\circ = \begin{bmatrix} \boldsymbol{U} & \\ & -\boldsymbol{U} \end{bmatrix}, \quad \boldsymbol{V}^\circ = \begin{bmatrix} \boldsymbol{V} & \\ & -\boldsymbol{V} \end{bmatrix}, \quad \boldsymbol{W}^\circ = \begin{bmatrix} \boldsymbol{W} & \\ & -\boldsymbol{W} \end{bmatrix} \tag{32}$$

$$\boldsymbol{T}_A = \begin{bmatrix} \boldsymbol{A} & \boldsymbol{A}_h & \begin{bmatrix} \boldsymbol{A}_h & \cdots & \boldsymbol{A}_h \end{bmatrix} & \boldsymbol{0} \end{bmatrix} \tag{33}$$

$$\boldsymbol{T}_I = \begin{bmatrix} \boldsymbol{I}_n & \boldsymbol{0} & \begin{bmatrix} \boldsymbol{0} & \cdots & \boldsymbol{0} \end{bmatrix} & \boldsymbol{0} \end{bmatrix} \tag{34}$$

$\boldsymbol{I}_n \in \mathbb{R}^{n\times n}$, $\boldsymbol{I}_{mn} \in \mathbb{R}^{mn\times mn}$ *are identity matrices,* $\boldsymbol{0} \in \mathbb{R}^{n\times n}$, $\boldsymbol{0}_w \in \mathbb{R}^{mn\times n}$ *are zero matrices, respectively, and* $\boldsymbol{U}^\circ, \boldsymbol{V}^\circ \in \mathbb{R}^{2n\times 2n}$, $\boldsymbol{W}^\circ \in \mathbb{R}^{2mn\times 2mn}$ *are structured matrix variables.*

Proof (compare [2], [6]) Defining Lyapunov-Krasovskii functional candidate as follows

$$\begin{aligned} v(\boldsymbol{q}(t)) &= \boldsymbol{q}^T(t)\boldsymbol{P}\boldsymbol{q}(t) + \int_{t-\frac{h}{m}}^{t} \boldsymbol{p}^T(s)\boldsymbol{W}\boldsymbol{p}(s)\mathrm{d}s + \\ &+ \int_{-\frac{h}{m}}^{0}\int_{t+\vartheta}^{t} \boldsymbol{q}^T(s)\boldsymbol{U}\boldsymbol{q}(s)\mathrm{d}s\mathrm{d}\vartheta + \int_{-\frac{h}{m}}^{0}\int_{\vartheta}^{0}\int_{t+\lambda}^{t} \dot{\boldsymbol{q}}^T(s)\boldsymbol{V}\dot{\boldsymbol{q}}(s)\mathrm{d}s\mathrm{d}\lambda\mathrm{d}\vartheta + \end{aligned} \tag{35}$$

with

$$\boldsymbol{p}^T(t) = \begin{bmatrix} \boldsymbol{p}_1^T(t) & \boldsymbol{p}_2^T(t) \end{bmatrix} \tag{36}$$

$$\boldsymbol{p}_1^T(t) = \int_{t-\frac{h}{m}}^{t} \boldsymbol{q}^T(s)\mathrm{d}s \tag{37}$$

$$\boldsymbol{p}_2^T(t) = \begin{bmatrix} \int_{t-\frac{2h}{m}}^{t-\frac{h}{m}} \boldsymbol{q}^T(s)\mathrm{d}s & \cdots & \int_{t-h}^{t-(m-1)\frac{h}{m}} \boldsymbol{q}^T(s)\mathrm{d}s \end{bmatrix} \tag{38}$$

and evaluating the derivative of $v(\boldsymbol{q}(t)$ along a solution of the autonomous system (1) it can be obtained

$$\dot{v}(\boldsymbol{q}(t)) = \dot{\boldsymbol{q}}^T(t)\boldsymbol{P}\boldsymbol{q}(t) + \boldsymbol{q}^T(t)\boldsymbol{P}\dot{\boldsymbol{q}}(t) + \dot{v}_1(\boldsymbol{q}(t)) + \dot{v}_2(\boldsymbol{q}(t)) + \dot{v}_3(\dot{\boldsymbol{q}}(t)) \tag{39}$$

where

$$\begin{aligned} \dot{v}_1(\boldsymbol{q}(t)) &= \frac{\mathrm{d}}{\mathrm{d}t}\Big(\int_{t-\frac{h}{m}}^{t} \boldsymbol{p}^T(s)\boldsymbol{W}\boldsymbol{p}(s)\mathrm{d}s\Big) = \\ &= \boldsymbol{p}^T(t)\boldsymbol{W}\boldsymbol{p}(t) - \boldsymbol{p}^T\big(t-\tfrac{h}{m}\big)\boldsymbol{W}\boldsymbol{p}\big(t-\tfrac{h}{m}\big) \end{aligned} \tag{40}$$

$$\begin{aligned} \dot{v}_2(\boldsymbol{q}(t)) &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{-\frac{h}{m}}^{0}\left\{\int_{t+\vartheta}^{t} \boldsymbol{q}^T(s)\boldsymbol{U}\boldsymbol{q}(s)\mathrm{d}s\right\}\mathrm{d}\vartheta\right) = \\ &= \int_{-\frac{h}{m}}^{0} \boldsymbol{q}^T(t)\boldsymbol{U}\boldsymbol{q}(t)\mathrm{d}\vartheta - \int_{-\frac{h}{m}}^{0} \boldsymbol{q}^T(t+\vartheta)\boldsymbol{U}\boldsymbol{q}(t+\vartheta)\mathrm{d}\vartheta = \\ &= \frac{h}{m}\boldsymbol{q}^T(t)\boldsymbol{U}\boldsymbol{q}(t) - \int_{t-\frac{h}{m}}^{t} \boldsymbol{q}^T(s)\boldsymbol{U}\boldsymbol{q}(s)\mathrm{d}s \end{aligned} \tag{41}$$

$$
\begin{aligned}
\dot{v}_3(\dot{\boldsymbol{q}}(t)) &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{-\frac{h}{m}}^{0}\int_{\vartheta}^{0}\left\{\int_{t+\lambda}^{t}\dot{\boldsymbol{q}}^T(s)\boldsymbol{V}\dot{\boldsymbol{q}}(s)\mathrm{d}s\right\}\mathrm{d}\lambda\mathrm{d}\vartheta\right) = \\
&= \int_{-\frac{h}{m}}^{0}\int_{\vartheta}^{0}\dot{\boldsymbol{q}}^T(t)\boldsymbol{V}\dot{\boldsymbol{q}}(t)\mathrm{d}\lambda\mathrm{d}\vartheta - \int_{-\frac{h}{m}}^{0}\int_{\vartheta}^{0}\dot{\boldsymbol{q}}^T(t+\lambda)\boldsymbol{V}\dot{\boldsymbol{q}}(t+\lambda)\mathrm{d}\lambda\mathrm{d}\vartheta = \\
&= \int_{-\frac{h}{m}}^{0} -\vartheta\dot{\boldsymbol{q}}^T(t)\boldsymbol{V}\dot{\boldsymbol{q}}(t)\mathrm{d}\vartheta - \int_{-\frac{h}{m}}^{0}\int_{t+\vartheta}^{t}\dot{\boldsymbol{q}}^T(s)\boldsymbol{V}\dot{\boldsymbol{q}}(s)\mathrm{d}s\mathrm{d}\vartheta = \\
&= \frac{1}{2}\left(\frac{h}{m}\right)^2\dot{\boldsymbol{q}}^T(t)\boldsymbol{V}\dot{\boldsymbol{q}}(t) - \int_{-\frac{h}{m}}^{0}\int_{t+\vartheta}^{t}\dot{\boldsymbol{q}}^T(s)\boldsymbol{V}\dot{\boldsymbol{q}}(s)\mathrm{d}s\mathrm{d}\vartheta
\end{aligned}
\tag{42}
$$

Subsequently, using (17), (37) then it yields

$$
\begin{aligned}
\dot{v}_2(\boldsymbol{q}(t)) &\leqslant \frac{h}{m}\boldsymbol{q}^T(t)\boldsymbol{U}\boldsymbol{q}(t) - \frac{m}{h}\int_{t-\frac{h}{m}}^{t}\boldsymbol{q}^T(s)\mathrm{d}s\,\boldsymbol{U}\int_{t-\frac{h}{m}}^{t}\boldsymbol{q}(s)\mathrm{d}s = \\
&= \frac{h}{m}\boldsymbol{q}^T(t)\boldsymbol{U}\boldsymbol{q}(t) - \frac{m}{h}\boldsymbol{p}_1^T(t)\boldsymbol{U}\boldsymbol{p}_1(t)
\end{aligned}
\tag{43}
$$

and analogously, using (16), (37) then it yields

$$
\begin{aligned}
\dot{v}_3(\dot{\boldsymbol{q}}(t)) &\leqslant \frac{1}{2}\left(\frac{h}{m}\right)^2\dot{\boldsymbol{q}}^T(t)\boldsymbol{V}\dot{\boldsymbol{q}}(t) - 2\left(\frac{m}{h}\right)^2\int_{-\frac{h}{m}}^{0}\int_{t+\vartheta}^{t}\dot{\boldsymbol{q}}^T(s)\mathrm{d}s\mathrm{d}\vartheta\,\boldsymbol{V}\int_{-\frac{h}{m}}^{0}\int_{t+\vartheta}^{t}\dot{\boldsymbol{q}}(s)\mathrm{d}s\mathrm{d}\vartheta = \\
&= \frac{1}{2}\left(\frac{h}{m}\right)^2\dot{\boldsymbol{q}}^T(t)\boldsymbol{V}\dot{\boldsymbol{q}}(t) - \\
&\quad -2\left(\frac{m}{h}\right)^2\int_{-\frac{h}{m}}^{0}\left(\boldsymbol{q}^T(t)-\boldsymbol{q}^T(t+\vartheta)\right)\mathrm{d}\vartheta\,\boldsymbol{V}\int_{-\frac{h}{m}}^{0}\Big(\boldsymbol{q}(t)-\boldsymbol{q}(t+\vartheta)\Big)\mathrm{d}\vartheta = \\
&= \frac{1}{2}\left(\frac{h}{m}\right)^2\dot{\boldsymbol{q}}^T(t)\boldsymbol{V}\dot{\boldsymbol{q}}(t) - \\
&\quad -2\left(\frac{m}{h}\right)^2\left(\frac{h}{m}\boldsymbol{q}^T(t) - \int_{t-\frac{h}{m}}^{t}\boldsymbol{q}^T(s)\mathrm{d}s\right)\boldsymbol{V}\left(\frac{h}{m}\boldsymbol{q}(t) - \int_{t-\frac{h}{m}}^{t}\boldsymbol{q}(s)\mathrm{d}s\right) = \\
&= \frac{1}{2}\left(\frac{h}{m}\right)^2\dot{\boldsymbol{q}}^T(t)\boldsymbol{V}\dot{\boldsymbol{q}}(t) - 2\left(\frac{m}{h}\right)^2\left(\frac{h}{m}\boldsymbol{q}^T(t)-\boldsymbol{p}_1^T(t)\right)\boldsymbol{V}\left(\frac{h}{m}\boldsymbol{q}(t)-\boldsymbol{p}_1(t)\right)
\end{aligned}
\tag{44}
$$

Since (36)–(38) implies

$$
\boldsymbol{p}^T(t-\frac{h}{m}) = \boldsymbol{p}_2^T(t) + \int_{t-h-\frac{h}{m}}^{t-h}\boldsymbol{q}^T(s)\mathrm{d}s \tag{45}
$$

defining the notations

$$
\boldsymbol{q}^{\circ T}(t) = \begin{bmatrix} \boldsymbol{q}^T(t) & \boldsymbol{p}_1^T(t) & \boldsymbol{p}_2^T(t) & \boldsymbol{p}_3^T(t) \end{bmatrix} \tag{46}
$$

$$\boldsymbol{p}_3^T(t) = \int_{t-h-\frac{h}{m}}^{t-h} \boldsymbol{q}^T(s)\mathrm{d}s \tag{47}$$

then with respect to (33), (34) it can be written for the parts of the autonomous system model

$$\boldsymbol{A}\boldsymbol{q}(t) + \boldsymbol{A}_h \int_{t-h}^{t} \boldsymbol{q}(s)\mathrm{d}s = \boldsymbol{T}_A\boldsymbol{q}^\circ(t) = \dot{\boldsymbol{q}}(t) \tag{48}$$

$$\boldsymbol{T}_I\boldsymbol{q}^\circ(t) = \boldsymbol{q}(t) \tag{49}$$

which implies for the first two elements of (39)

$$\dot{\boldsymbol{q}}^T(t)\boldsymbol{P}\boldsymbol{q}(t) + \boldsymbol{q}^T(t)\boldsymbol{P}\dot{\boldsymbol{q}}(t) = \boldsymbol{q}^{\circ T}(t)(\boldsymbol{T}_A^T\boldsymbol{P}\boldsymbol{T}_I + \boldsymbol{T}_I^T\boldsymbol{P}\boldsymbol{T}_A)\boldsymbol{q}^\circ(t) \tag{50}$$

In the same sense, using (28)–(32) it can be obtained

$$\dot{v}_1(\boldsymbol{q}(t)) = \boldsymbol{p}^T(t)\boldsymbol{W}\boldsymbol{p}(t) - \boldsymbol{p}^T(t-\frac{h}{m})\boldsymbol{W}\boldsymbol{p}(t-\frac{h}{m}) = \boldsymbol{q}^{\circ T}(t)\boldsymbol{T}_W^T\boldsymbol{W}^\circ\boldsymbol{T}_W\boldsymbol{q}^\circ(t) \tag{51}$$

$$\dot{v}_2(\boldsymbol{q}(t)) \leqslant \frac{h}{m}\boldsymbol{q}^T(t)\boldsymbol{U}\boldsymbol{q}(t) - \frac{m}{h}\boldsymbol{p}_1^T(t)\boldsymbol{U}\boldsymbol{p}_1(t) = \boldsymbol{q}^{\circ T}(t)\boldsymbol{T}_U^T\boldsymbol{U}^\circ\boldsymbol{T}_U\boldsymbol{q}^\circ(t) \tag{52}$$

$$\begin{aligned}\dot{v}_3(\dot{\boldsymbol{q}}(t)) \leqslant \tfrac{1}{2}\left(\tfrac{h}{m}\right)^2\dot{\boldsymbol{q}}^T(t)\boldsymbol{V}\dot{\boldsymbol{q}}(t) - 2\left(\tfrac{m}{h}\right)^2(\tfrac{h}{m}\boldsymbol{q}^T(t) - \boldsymbol{p}_1^T(t))\boldsymbol{V}(\tfrac{h}{m}\boldsymbol{q}(t) - \boldsymbol{p}_1(t)) = \\ = \boldsymbol{q}^{\circ T}(t)\boldsymbol{T}_V^T\boldsymbol{V}^\circ\boldsymbol{T}_V\boldsymbol{q}^\circ(t)\end{aligned} \tag{53}$$

With $\boldsymbol{P}^\circ$ defined in (27) it follows that

$$\dot{v}(\boldsymbol{q}(t)) \leqslant \boldsymbol{q}^{\circ T}(t)\boldsymbol{P}^\circ\boldsymbol{q}^\circ(t) < 0 \tag{54}$$

if the matrix inequality $\boldsymbol{P}^\circ < 0$ is feasible, i.e. (54) implies (27). This concludes the proof. □

5. Control law parameter design

With the preceded results, considering the control law parameter $\boldsymbol{K}$, it is now possible to state the next design conditions.

Theorem 5 *The closed-loop system* (1) *controlled by the control law* (4) *is asymptotically stable if for given $h > 0$, $m > 0$ there exist symmetric positive definite matrices $\boldsymbol{Y}, \boldsymbol{U}^\bullet, \boldsymbol{V}^\bullet \in \mathbb{R}^{n\times n}$, $\boldsymbol{W}^\bullet \in \mathbb{R}^{mn\times mn}$, and a matrix $\boldsymbol{Z} \in \mathbb{R}^{r\times n}$ such that*

$$\boldsymbol{Y} = \boldsymbol{Y}^T > 0, \quad \boldsymbol{U}^\bullet = \boldsymbol{U}^{\bullet T} > 0, \quad \boldsymbol{V}^\bullet = \boldsymbol{V}^{\bullet T} > 0, \quad \boldsymbol{W}^\bullet = \boldsymbol{W}^{\bullet T} > 0 \tag{55}$$

$$\begin{bmatrix} \boldsymbol{P}^{\diamond} & * \\ \boldsymbol{T}_A^{\diamond}\boldsymbol{Y}^{\diamond} & -b^{-2}\boldsymbol{V}^{\bullet} \end{bmatrix} < 0 \tag{56}$$

$$\boldsymbol{P}^{\diamond} = \boldsymbol{Y}^{\diamond T}\boldsymbol{T}_A^{\diamond T}\boldsymbol{T}_I + \boldsymbol{T}_I^T\boldsymbol{T}_A^{\diamond}\boldsymbol{Y}^{\diamond} + \boldsymbol{T}_U^T\boldsymbol{U}^{\diamond}\boldsymbol{T}_U + \boldsymbol{T}_{V2}^T\boldsymbol{V}_2^{\diamond}\boldsymbol{T}_{V2} + \boldsymbol{T}_W^T\boldsymbol{W}^{\diamond}\boldsymbol{T}_W \tag{57}$$

where

$$\boldsymbol{U}^{\diamond} = \begin{bmatrix} \boldsymbol{U}^{\bullet} & \\ & -\boldsymbol{U}^{\bullet} \end{bmatrix}, \quad \boldsymbol{W}^{\diamond} = \begin{bmatrix} \boldsymbol{W}^{\bullet} & \\ & -\boldsymbol{W}^{\bullet} \end{bmatrix}, \quad \boldsymbol{V}_2^{\diamond} = \boldsymbol{V}^{\bullet} - 2\boldsymbol{Y} \tag{58}$$

$$\boldsymbol{Y}^{\diamond} = diag\begin{bmatrix} \begin{bmatrix} \boldsymbol{Y} \\ \boldsymbol{Z} \end{bmatrix} & \boldsymbol{Y} & diag\begin{bmatrix} \boldsymbol{Y} & \cdots & \boldsymbol{Y} \end{bmatrix} & \boldsymbol{Y} \end{bmatrix} \tag{59}$$

$\boldsymbol{Y}^{\diamond} \in \mathbb{R}^{(n(m+2)+r)\times n(m+2)}$, $\boldsymbol{W}^{\diamond} \in \mathbb{R}^{2rn\times 2rn}$, $\boldsymbol{U}^{\circ} \in \mathbb{R}^{2n\times 2n}$ *are structured matrix variables,*

$$\boldsymbol{T}_{V2} = b^{-1}\begin{bmatrix} a^2\boldsymbol{I}_n & -\boldsymbol{I}_n & \begin{bmatrix} \boldsymbol{0} & \cdots & \boldsymbol{0} \end{bmatrix} & \boldsymbol{0} \end{bmatrix} \tag{60}$$

$$\boldsymbol{T}_A^{\diamond} = \begin{bmatrix} [\boldsymbol{A} \ \ -\boldsymbol{B}] & \boldsymbol{A}_h & \begin{bmatrix} \boldsymbol{A}_h & \cdots & \boldsymbol{A}_h \end{bmatrix} & \boldsymbol{0} \end{bmatrix} \tag{61}$$

and $\boldsymbol{T}_U$, $\boldsymbol{T}_W$, $\boldsymbol{T}_I$, a and b are as in (28), (30), (34), (31), respectively.

Now, the control gain matrix $\boldsymbol{K}$ can be found directly as

$$\boldsymbol{K} = \boldsymbol{Z}\boldsymbol{Y}^{-1} \tag{62}$$

Hereafter, $*$ denotes the symmetric item in a symmetric matrix.

Proof By Schur complement (9) the inequality (27) is equivalent to

$$\begin{bmatrix} \boldsymbol{P}^{\bullet} & \boldsymbol{T}_{V1}^T \\ \boldsymbol{T}_{V1} & -b^{-2}\boldsymbol{V}^{-1} \end{bmatrix} = \begin{bmatrix} \boldsymbol{P}^{\bullet} & \boldsymbol{T}_A^T \\ \boldsymbol{T}_A & -b^{-2}\boldsymbol{V}^{-1} \end{bmatrix} < 0 \tag{63}$$

where $\boldsymbol{T}_{V1}$, $\boldsymbol{T}_{V2}$ are the first and the second row of $\boldsymbol{T}_V$, respectively and

$$\boldsymbol{P}^{\bullet} = \boldsymbol{T}_A^T\boldsymbol{P}\boldsymbol{T}_I + \boldsymbol{T}_I^T\boldsymbol{P}\boldsymbol{T}_A + \boldsymbol{T}_U^T\boldsymbol{U}^{\diamond}\boldsymbol{T}_U + \boldsymbol{T}_W^T\boldsymbol{W}^{\diamond}\boldsymbol{T}_W - \boldsymbol{T}_{V2}^T\boldsymbol{V}\boldsymbol{T}_{V2} \tag{64}$$

Then defining the congruence transform matrix

$$\boldsymbol{T}_c = \text{diag}\begin{bmatrix} \boldsymbol{T}_{c1} & \boldsymbol{I}_n \end{bmatrix} = \text{diag}\begin{bmatrix} \boldsymbol{P}^{-1} & \boldsymbol{P}^{-1} & \text{diag}\begin{bmatrix} \boldsymbol{P}^{-1} & \cdots & \boldsymbol{P}^{-1} \end{bmatrix} & \boldsymbol{P}^{-1} & \boldsymbol{I}_n \end{bmatrix} \tag{65}$$

$$\boldsymbol{T}_{c1} = \text{diag}\begin{bmatrix} \boldsymbol{P}^{-1} & \boldsymbol{P}^{-1} & \text{diag}\begin{bmatrix} \boldsymbol{P}^{-1} & \cdots & \boldsymbol{P}^{-1} \end{bmatrix} & \boldsymbol{P}^{-1} \end{bmatrix} \tag{66}$$

and pre-multiplying right-hand side and left-hand side of (63) by (65) gives the next result

$$\begin{bmatrix} \boldsymbol{T}_{c1}\boldsymbol{P}^{\bullet}\boldsymbol{T}_{c1} & \boldsymbol{T}_{c1}\boldsymbol{T}_A^T \\ \boldsymbol{T}_A\boldsymbol{T}_{c1} & -b^{-2}\boldsymbol{V}^{-1} \end{bmatrix} < 0 \tag{67}$$

Defining $\boldsymbol{P}^{-1} = \boldsymbol{Y}$ then

$$\boldsymbol{T}_A\boldsymbol{T}_{c1} = \boldsymbol{T}_A\boldsymbol{Y}^{\bullet} \tag{68}$$

$$\boldsymbol{Y}^{\bullet} = \boldsymbol{T}_{c1} = \operatorname{diag}\begin{bmatrix} \boldsymbol{Y} & \boldsymbol{Y} & \operatorname{diag}\begin{bmatrix} \boldsymbol{Y} & \cdots & \boldsymbol{Y} \end{bmatrix} & \boldsymbol{Y} \end{bmatrix} \tag{69}$$

Subsequently, with the notation (58), it yields

$$\boldsymbol{T}_{c1}\left(\boldsymbol{T}_A^T\boldsymbol{P}\boldsymbol{T}_I + \boldsymbol{T}_I^T\boldsymbol{P}\boldsymbol{T}_A\right)\boldsymbol{T}_{c1} = \boldsymbol{Y}^{\bullet}\boldsymbol{T}_A^T\boldsymbol{T}_I + \boldsymbol{T}_I^T\boldsymbol{T}_A\boldsymbol{Y}^{\bullet} \tag{70}$$

$$\boldsymbol{T}_{c1}\boldsymbol{T}_U^T\boldsymbol{U}^{\circ}\boldsymbol{T}_U\boldsymbol{T}_{c1} = \boldsymbol{T}_U^T\boldsymbol{U}^{\diamond}\boldsymbol{T}_U, \tag{71}$$

$$\boldsymbol{U}^{\diamond} = \begin{bmatrix} \boldsymbol{U}^{\bullet} & \\ & -\boldsymbol{U}^{\bullet} \end{bmatrix}, \qquad \boldsymbol{U}^{\bullet} = \boldsymbol{Y}\boldsymbol{U}\boldsymbol{Y} \tag{72}$$

$$\boldsymbol{T}_{c1}\boldsymbol{T}_W^T\boldsymbol{W}^{\circ}\boldsymbol{T}_W\boldsymbol{T}_{c1} = \boldsymbol{T}_W^T\boldsymbol{W}^{\diamond}\boldsymbol{T}_W \tag{73}$$

$$\boldsymbol{W}^{\diamond} = \begin{bmatrix} \boldsymbol{W}^{\bullet} & \\ & -\boldsymbol{W}^{\bullet} \end{bmatrix}, \quad \boldsymbol{W}^{\bullet} = \operatorname{diag}\begin{bmatrix} \boldsymbol{Y} & \cdots & \boldsymbol{Y} \end{bmatrix}\boldsymbol{W}\operatorname{diag}\begin{bmatrix} \boldsymbol{Y} & \cdots & \boldsymbol{Y} \end{bmatrix} \tag{74}$$

and with (6)

$$\boldsymbol{T}_{c1}\boldsymbol{T}_{V2}^T\boldsymbol{V}\boldsymbol{T}_{V2}\boldsymbol{T}_{c1} = \boldsymbol{T}_{V2}^T\boldsymbol{P}^{-1}\boldsymbol{V}\boldsymbol{P}^{-1}\boldsymbol{T}_{V2} \geqslant \boldsymbol{T}_{V2}^T\left(2\boldsymbol{P}^{-1} - \boldsymbol{V}^{-1}\right)\boldsymbol{T}_{V2} \tag{75}$$

$$-\boldsymbol{T}_{c1}\boldsymbol{T}_{V2}^T\boldsymbol{V}\boldsymbol{T}_{V2}\boldsymbol{T}_{c1} \leqslant \boldsymbol{T}_{V2}^T\boldsymbol{V}_2^{\diamond}\boldsymbol{T}_{V2} \tag{76}$$

respectively, where

$$\boldsymbol{V}^{-1} = \boldsymbol{V}^{\bullet}, \qquad \boldsymbol{P}^{-1} = \boldsymbol{Y}, \qquad \boldsymbol{V}_2^{\diamond} = \boldsymbol{V}^{\bullet} - 2\boldsymbol{Y}. \tag{77}$$

Thus, (67), (68) can be rewritten as

$$\begin{bmatrix} \boldsymbol{P}_A^{\bullet} & \boldsymbol{Y}^{\bullet T}\boldsymbol{T}_A^T \\ \boldsymbol{T}_A\boldsymbol{Y}^{\bullet} & -b^{-2}\boldsymbol{V}^{\bullet} \end{bmatrix} < 0 \tag{78}$$

$$\boldsymbol{P}_A^{\bullet} = \boldsymbol{Y}^{\bullet}\boldsymbol{T}_A^T\boldsymbol{T}_I + \boldsymbol{T}_I^T\boldsymbol{T}_A\boldsymbol{Y}^{\bullet} + \boldsymbol{T}_U^T\boldsymbol{U}^{\circ}\boldsymbol{T}_U + \boldsymbol{T}_W^T\boldsymbol{W}^{\circ}\boldsymbol{T}_W + \boldsymbol{T}_{V2}^T\boldsymbol{V}_2^{\circ}\boldsymbol{T}_{V2} \tag{79}$$

Replacing the matrix $\boldsymbol{A}$ in (33) by the closed-loop system matrix $\boldsymbol{A}_c = \boldsymbol{A} - \boldsymbol{B}\boldsymbol{K}$ results in

$$\boldsymbol{A}_c\boldsymbol{Y} = \boldsymbol{A}\boldsymbol{Y} - \boldsymbol{B}\boldsymbol{K}\boldsymbol{Y} \tag{80}$$

and defining

$$\boldsymbol{K}\boldsymbol{Y} = \boldsymbol{Z} \tag{81}$$

it can be written

$$(\boldsymbol{T}_A\boldsymbol{Y}^{\bullet})_{A_c} = \boldsymbol{T}_A^{\diamond}\boldsymbol{Y}^{\diamond} \tag{82}$$

$$(\boldsymbol{Y}^{\bullet}\boldsymbol{T}_A^T\boldsymbol{T}_I + \boldsymbol{T}_I^T\boldsymbol{T}_A\boldsymbol{Y}^{\bullet})_{A_c} = \boldsymbol{Y}^{\diamond}\boldsymbol{T}_A^{\circ T}\boldsymbol{T}_I + \boldsymbol{T}_I^T\boldsymbol{T}_A^{\circ}\boldsymbol{Y}^{\diamond}, \qquad \boldsymbol{P}_{A_c}^{\bullet} = \boldsymbol{P}^{\diamond} \tag{83}$$

owing to that with (59), (61) it is

$$\begin{bmatrix} \boldsymbol{A}-\boldsymbol{BK} & \boldsymbol{A}_h & \begin{bmatrix} \boldsymbol{A}_h & \cdots & \boldsymbol{A}_h \end{bmatrix} & \boldsymbol{0} \end{bmatrix} \operatorname{diag}\begin{bmatrix} \boldsymbol{Y} & \boldsymbol{Y} & \operatorname{diag}\begin{bmatrix} \boldsymbol{Y} & \cdots & \boldsymbol{Y} \end{bmatrix} & \boldsymbol{Y} \end{bmatrix} =$$
$$= \begin{bmatrix} [\boldsymbol{A} \;\; -\boldsymbol{B}] & \boldsymbol{A}_h & \begin{bmatrix} \boldsymbol{A}_h & \cdots & \boldsymbol{A}_h \end{bmatrix} & \boldsymbol{0} \end{bmatrix} \operatorname{diag}\begin{bmatrix} \begin{bmatrix} \boldsymbol{Y} \\ \boldsymbol{Z} \end{bmatrix} & \boldsymbol{Y} & \operatorname{diag}\begin{bmatrix} \boldsymbol{Y} & \cdots & \boldsymbol{Y} \end{bmatrix} & \boldsymbol{Y} \end{bmatrix} \tag{84}$$

Thus, replacing $\boldsymbol{T}_A\boldsymbol{Y}^{\bullet}$ by $\boldsymbol{T}_A^{\diamond}\boldsymbol{Y}^{\diamond}$ in (78), (79) these implies (56), (57). This concludes the proof. □

6. Illustrative example

To demonstrate the algorithm properties it was assumed that system is given by (1), (2), where the system parameters are

$$\boldsymbol{A} = \begin{bmatrix} 2.6 & 0.0 & -0.8 \\ 1.2 & 0.2 & 0.0 \\ 0.0 & -0.5 & 3.0 \end{bmatrix}, \; \boldsymbol{C} = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$$\boldsymbol{A}_h = \begin{bmatrix} 0.00 & 0.02 & 0.00 \\ 0.00 & 0.00 & -1.00 \\ -0.02 & 0.00 & 0.00 \end{bmatrix}, \; \boldsymbol{B} = \begin{bmatrix} 1 & 3 \\ 2 & 1 \\ 1 & 1 \end{bmatrix}$$

Setting $m = 2$ and solving (55), (56) with respect the LMI matrix variables $\boldsymbol{Y}$, $\boldsymbol{Z}$, $\boldsymbol{U}^{\bullet}$, $\boldsymbol{V}^{\bullet}$, and $\boldsymbol{W}^{\bullet}$ using Self-Dual-Minimization (SeDuMi) package [15] for Matlab, the gain matrix problem was solved as feasible for $h \leqslant 5.2$ s, with

$$\boldsymbol{K} = \begin{bmatrix} -3.8810 & -2.3846 & 14.2883 \\ 4.1853 & 2.1543 & -11.5520 \end{bmatrix}, \; \boldsymbol{A}_c = \begin{bmatrix} -6.0749 & -4.0782 & 19.5678 \\ 4.7767 & 2.8150 & -17.0246 \\ -0.3043 & -0.2697 & 0.2637 \end{bmatrix}$$

and with the stable eigenvalue spectrum of the closed-loop system matrix $\operatorname{eig}(\boldsymbol{A}_c) = \{-1.6148,\ -0.6907 \pm 0.4192\mathrm{i}\}$.

To characterize the steady-state control properties the extended closed-loop system matrix $\boldsymbol{A}_{ce} = \boldsymbol{A} + \boldsymbol{A}_h - \boldsymbol{BK}$ was computed, where

$$\boldsymbol{A}_{ce} = \begin{bmatrix} -6.0749 & -4.0582 & 19.5678 \\ 4.7767 & 2.8150 & -18.0246 \\ -0.3243 & -0.2697 & 0.2637 \end{bmatrix}, \; \operatorname{eig}(\boldsymbol{A}_{ce}) = \{-1.3692,\ -0.8135 \pm 0.1423\mathrm{i}\}$$

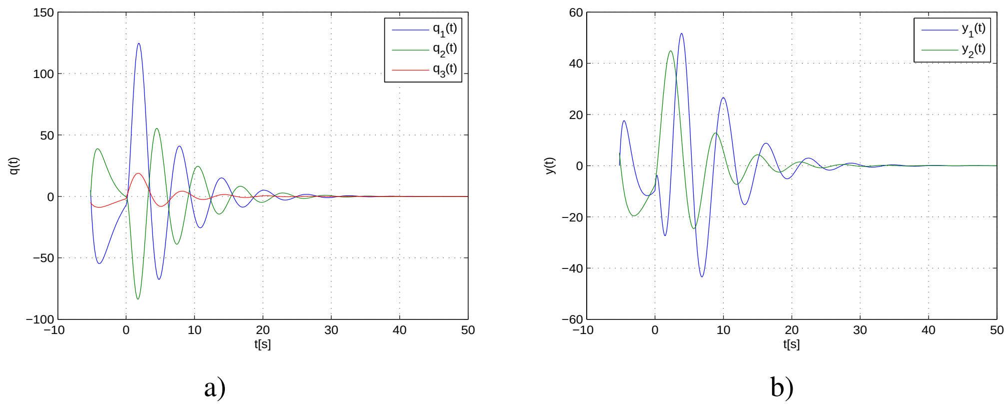

Figure 1. Responses of the system: a) state variables, b) output variables

Setting initial state as $\boldsymbol{q}_h^T(-5.2) = [0.1\ 0.0\ -0.1]$, Fig. 1 shows the closed-loop system state and output response.

Note, presented algorithms is enough robust to the system time-delay value h in that sense that for given m there exists such upper bound of h the design task be feasible. It is possible to verify that e.g. if $m = 3$ then $h \leqslant 6.7s$, if $m = 4$ then $h \leqslant 8.1s$, etc.

7. Concluding remarks

Design conditions, explained with respect to formal limitations triggered by existence of structured matrix variables in LMIs, and formulated using an extended version of the Lyapunov-Krasovskii functional, are derived in the paper. Obtained formulation is a convex LMI problem where the manipulation is accomplished in that manner that produces the closed-loop system asymptotical stability. Presented illustrative example confirms the effectiveness of proposed control design techniques. In particular, with the use of an extended version of Lyapunov-Krasovskii functional, it was shown how to adapt the standard approach to design optimal matrix parameters of state controllers for linear systems with distributed time delays.

References

[1] D. BOYD, L. EL GHAOUI, E. PERON and V. BALAKRISHNAN: Linear Matrix Inequalities in System and Control Theory. Philadelphia, SIAM Society for Industrial and Applied Mathematics, 1994.

[2] Z. FENG and J. LAM: Integral partitioning approach to stability analysis and sta-

bilization of distributed time delay systems. *Preprints of the 18th IFAC World Congress*, Milano, Italy, (2011), 5094-5099.

[3] Z. FENG and J. LAM: Integral partitioning approach to robust stabilization for uncertain distributed time delay systems. *Int. J. of Robust and Nonlinear Control*, **22** (2012), 676-689.

[4] Y.A. FIAGBEDZI and A.E. PEARSON: A multistage reduction technique for feedback stabilizing distributed time-lag systems. *Automatica*, *23*(3), (1987), 311-326.

[5] P. GAHINET, A. NEMIROVSKI, A.J. LAUB and M. CHILALI: LMI Control Toolbox User's Guide. Natick, The MathWorks, Inc., 1995.

[6] D. GONTKOVIČ and R. FÓNOD: State control design for linear systems with distributed time delay. *Proc. of the 10th IEEE Jubilee Int. Symp. on Applied Machine Intelligence and Informatics SAMI 2012*, Herĺany, Slovakia, (2012), 97-101.

[7] F. GOUAISBAUT and D. PEAUCELLE: Delay-dependent stability analysis of linear time delay systems. *Proc. of the 6th IFAC Workshop on Time Delay System TDS'06*, LAquila, Italy, (2006).

[8] K. GU: An improved stability criterion for systems with distributed delays. *Int. J. of Robust and Nonlinear Control*, **13** (2003), 819-831.

[9] G. HERRMANN, M.C. TURNER and I. POSTLETHWAITE: Linear matrix inequalities in control. Mathematical Methods for Robust and Nonlinear Control. Berlin, Springer-Verlag, 2007, 123-142.

[10] N.N. KRASOVSKII: On the application of Lyapunov's second method for equations with time delays. *Prikladnaja matematika i mechanika*, **20** (1956), 315-327, (in Russian).

[11] N.N. KRASOVSKII: Stability of Motion: Application of Lyapunov's Second Method to Differential Systems and Equations with Delay. Standford, Standford University Press, 1963.

[12] D. KROKAVEC and A. FILASOVÁ: Discrete-Time Systems. Košice, Elfa, 2006, (in Slovak).

[13] D. KROKAVEC and A. FILASOVÁ: Exponential stability of networked control systems with network-induced random delays. *Archives of Control Sciences*, **20**(2), (2010), 165-186.

[14] S.I. NICULESCU, E.I. VERIEST, L. DUGARD and J.M. DION: Stability and robust stability of time-delay systems: A guided tour. Stability and Control of Time-delay Systems. Berlin, Springer-Verlag, 1998, 1-71.

[15] D. PEAUCELLE, D. HENRION, Y. LABIT and K. TAITZ: User's Guide for SeDuMi Interface 1.04. Toulouse, LAAS-CNRS, 2002.

[16] U. SHAKED, I. YAESH and C.E. DE SOUZA: Bounded real criteria for linear time systems with state-delay. *IEEE Trans. on Automatic Control*, **43** (1998), 1116-1121.

[17] Y.S. SUH, H.J. KANG and Y.S. RO: Stability condition of distributed delay systems based on an analytic solution to Lyapunov functional equations. *Asian J. of Control*, **8** (2006), 91-96.

[18] J. SUN, J. CHEN, G. LIU and D. REES: On robust stability of uncertain neutral systems with discrete and distributed delays. *Proc. of American Control Conference*, St. Louis, MO, USA, (2009), 5469-5473.

[19] M. WU, Y. HE, J.J. SHE and G.P. LIU: Delay-dependent criteria for robust stability of time-varying delay systems. *Automatica*, **40** (2004), 1435-1439.

[20] F. ZHENG and P.M. FRANK: Robust control of uncertain distributed delay systems with application to the stabilization of combustion in rocket motor chambers. *Automatica*, **38** (2002), 487-497.

[21] Q.C. ZHONG: Robust Control of Time-delay Systems. London, Sprin-ger-Verlag, 2006.