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Application of normal possibility decision rule to silence

by

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Abstract: We often fall into silence. This often happens when we have two conflicting objectives and utilities, related to "cool head" and "warm heart". This case has two different states of nature, associated with "cool head" and "warm heart". We try to fuse the two decision problems referring to these different states of nature by introducing two-dimensional fuzzy events based on "cool head" and "warm heart". We construct a decision rule based on one-dimensional fuzzy events. Thus, we propose the normal possibility decision rule based on the normal possibility theory. In the example of this paper, we consider fuzzy events named "astray" state and "lost" state, related to the "cool head" and "warm heart". We can obtain the fuzzy utility functions by the extension principle for a mapping, and the fuzzy expected utility functions by the extension principle for the sum and the product. We assume that the DM (decision-maker) defines the weights for the individual states of nature and the two problems. We make full use of these weights and the fuzzy utility functions are transformed into the one-dimensional function. As we make full use of indexes for ordering of fuzzy numbers, we can order the weighted fuzzy expected utility and select the optimal decision. For the example of this paper, we assume that the possibility of a fuzzy event is normal possibility distributed, and a DM is risk neutral. Consequently, both any fuzzy utility function and any fuzzy expected possibility function are normal possibility distributed. A decision rule is introduced, based on the ordering of only means of these normal possibility distributions for the fuzzy expected utilities, so that we do not need an index for ordering. When DM is of another type, the fuzzy expected possibility function is in general not normally possibility distributed. In this case, the DM needs the indexes for the ordering of the fuzzy numbers. This fuzzy-Bayes decision rule provides for a natural extension of the scope of our study by increasing the dimension of the possibility function of a fuzzy event.

Keywords: fuzzy event, normal possibility theory, two compet-

1. Introduction

We often fall into silence. In the decision process usually involved therein we deal with two conflicting objectives and utilities, related to what we can call "cool head" and "warm heart" (see Fig. 1). This situation has two different states of

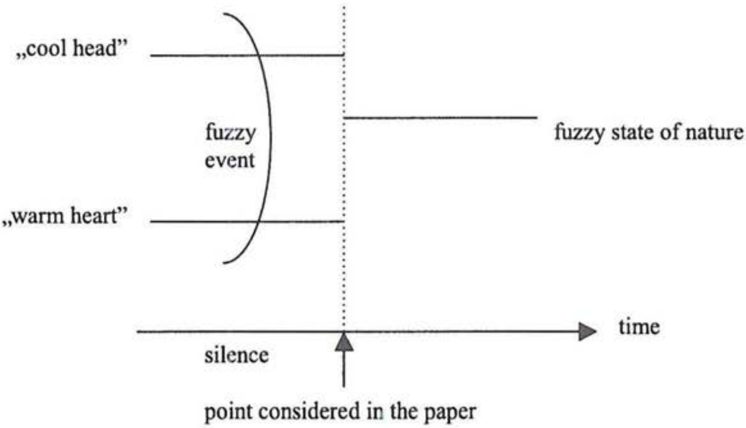

Figure 1. Schematic illustration of the situation considered in the paper

nature (variables), associated with "cool head" and "warm heart". We try to fuse the two decision problems referring to these different states of nature by introducing the two-dimensional fuzzy events based on "cool head" and "warm heart". We construct a decision rule based on one-dimensional fuzzy events (Uemura, 1991, 1993). Thus, we propose the normal possibility decision rule (Uemura, 1996) based on the normal possibility theory (Tanaka and Ishibuchi, 1992). In the example of this paper, we consider fuzzy events named the "astray" state and the "lost" state, related to the "cool head" and "warm heart". We can obtain the fuzzy utility functions by the extension principle for a mapping, and the fuzzy expected utility functions by the extension principle for the sum and the product. We assume the DM (decision-maker) can obtain the weights for the individual states of nature and the two problems. We make full use of these weights and the fuzzy utility functions are transformed into the one-dimensional function. As we make full use of indexes for ordering of fuzzy numbers, we can order the weighted fuzzy expected utility and select the optimal decision. For the example of this paper, we assume that the possibility of a fuzzy event is normal possibility distributed, and a DM is risk neutral. Consequently, both any fuzzy utility function and any fuzzy expected possibility function are normal possibility distributed. A decision rule is introduced, based on the ordering utilities (Uemura, 1996), so that we do not need an index for ordering. When DM is of another type, the fuzzy expected possibility function is in general not normally possibility distributed. In this case, the DM needs the indexes for the ordering of the fuzzy numbers. This fuzzy-Bayes decision rule provides for a natural extension of the scope of our study by increasing the dimension of the possibility function of a fuzzy event.

2. The fuzzy-Bayes decision rule for fuzzy events

We assume two states of nature related to two competing decision problems and denote them A_1 and A_2. In this paper we will further assume that the decision maker (DM) has the common decisions D_i $(i = 1, \ldots, m)$ pertaining to the two-decision problem. We will refer to the problem portrayed as the "cool head" problem through no. 1, while to that of the "warm heart"—through no. 2. We assume that the DM selects the same dimensionality of the two states of nature in the two problems (possibility of application of the same decisions). When the DM selects the different ones, we will assume that the DM can ignore the little important aspects of the states of nature and obtain the same number of the states of nature in the two competing problems. Now, problem no. 1 is $\langle s_1, D_i, U_{(1,D_i)}, \pi_1 \rangle$, and problem no. 2 is $\langle s_2, D_i, U_{(2,D_i)}, \pi_2 \rangle$, where $\pi_1(s_1)$, $\pi_2(s_2)$ are the prior possibility distributions over the states of nature $s_{1,2} \in R^n$, and $U_{(1,D_i)}$, $U_{(2,D_i)}$ are the utility functions related to the two states of nature and the decisions D_i, both normalised to $[0, 1]$.

Now, s/he decides on the importance weights for the two states of nature, denoted w_1 and $w_2 \in R^n$, and on the importance weights for the problems 1 and 2, denoted a and b. Then, s/he considers the two- dimensional fuzzy events, the dimensions corresponding to the indices of the states of nature, portrayed here as "cool head" and "warm heart", for instance—the "astray" and the "lost" states. The fuzzy events are denoted F_j $(j = 1, \ldots, l)$. The associated fuzzy decision problem is $\langle F, D_i, U_F, \Pi_F \rangle$, where U_F is a fuzzy utility function, and Π_F is a possibility measure for a fuzzy event. We assume the possibility function $\mu_{F_j}(s_1, s_2)$ for the fuzzy events F_j. In this case we obtain a possibility measure for a fuzzy event in the following form (Uemura, 1996), where the fuzzy events are related to the respective states of nature via the appropriate membership functions

a possibility measure for a fuzzy event

$$\Pi_{F_j} = \max_{s_1} \max_{s_2} [\mu_{F_j}(s_1, s_2) \cdot \pi(s_1, s_2)] \tag{1}$$

$$= \max_{s_1} \max_{s_2} [\mu_{F_j}(s_1, s_2) \cdot \pi(s_1) \cdot \pi_2(s_2)] \tag{2}$$

where $\pi(s_1, s_2) = \pi(s_1) \cdot \pi(s_2)$, since s_1 and s_2 are assumed to be independent.

We obtain in this situation the fuzzy utility function by application of the

a fuzzy utility function

$$U_{F_j,D_i}(z) = \int \mu_{F_j}(s_1, s_2)/[U_{(D_i)}(s_1, s_2)] \tag{3}$$

$$= \int \mu_{F_j}(s_1) \wedge \mu_{F_j}(s_2)/[U_{(D_i)}(s_1, s_2)] \tag{4}$$

$$= \sup_{\{s_1,s_2 | z=\mu_{F_j}(s_2)\}} [U_{(1,D_i)}(s_1) \wedge U_{(2,D_i)}(s_2)] \tag{5}$$

$$= \sup_{\{s_1 | z_1=\mu_{F_j}(s_1)\}} [U_{(1,D_i)}(s_1)] \wedge \sup_{\{s_2 | z_2=\mu_{F_j}(s_2)\}} [U_{(2,D_i)}(s_2)] \tag{6}$$

$$= \mu_{F_j}(U^{-1}_{(1,D_i)}(z_1)) \wedge \mu_{F_j}(U^{-1}_{(2,D_i)}(z_2)) \tag{7}$$

$$= \mu_{F_j}(U^{-1}_{(1,D_i)}(z_1), \mu_{F_j}(U^{-1}_{(2,D_i)}(z_2)) \tag{8}$$

where we remind that s_1 and s_2 are independent, $z_1, z_2 \in R^n$, $z \in R^{2n}$, $U_{(D_i)}(s_1, s_2)$ is the multiple utility function, and $U^{-1}_{(1,D_i)}(z_1)$, $U^{-1}_{(2,D_i)}(z_2)$ are inverse functions of the utility functions. Note that the utility function is understood to be the same as the possibility function when it is normalised to $[0, 1]$, and $U_{(D_i)}(s_1, s_2) = U_{(1,D_i)}(s_1) \wedge U_{(2,D_i)}(s_2)$. We obtain the fuzzy expected utility function by the extension principle for the sum and the product as follows (Uemura, 1996):

$$E_{D_i}(z) = \Pi_{F_l} \otimes U_{(F_l,D_i)}(z) \oplus \ldots \oplus \Pi_{F_l} \otimes U_{(F_l,D_i)}(z) \tag{9}$$

where $\oplus$ is the bounded sum, and $\otimes$ is the bounded product.

Now we set the weight vector $[aw_1, bw_2]^t$, in accordance with the previous notations, $a, b \in R^1$, $w_1, w_2 \in R^n$, and $w \in R^{2n}$. Further, we set $EE_{D_i}(zz) = w \cdot E_{D_i}(z)$. Note that $zz \in R^1$.

The decision rule that we construct makes full use of the index for the ordering of fuzzy numbers, proposed by Dubois and Prade (1988). This index is as follows:

$$Pos(E_{D_l} > E_{D_k}) = \sup_{y} \inf_{x \geq y} \min(E_{D_l}(x), 1 - E_{D_k}(y)) \tag{10}$$

Fuzzy decision making is based on $\max(Pos(E_{D_l} > E_{D_k}), Pos(E_{D_l} > E_{D_m}))$. We obtained a decision rule that serves to choose an optimal decision D_l for which we have $Pos(E_{D_l} > E_{D_k}) \geq 0.5$ $(k = 1, \ldots, m)$, see Uemura (1991).

3. Application of the fuzzy-Bayes decision rule to the normal possibility theory

We assume that the DM is risk neutral. S/he obtains the utility functions related to the "cool head" and "warm heart" by the certainty method from Keeney and

$$U_1(s_1, D_i) = c_{1i}s_1 + e_{1i} \tag{11}$$
$$U_2(s_2, D_i) = c_{2i}s_2 + e_{2i} \tag{12}$$

where $c_{1i} \in R^n$, $e_{1i} \in R^n$ $(i = 1, \ldots, m)$.

Now, s/he sets the normal possibility function of the fuzzy events $\mu_{F_j}(s_i) = (m_{ij}, D_{ij}^{-1})_e$, where $m_{ij} \in R^n$, and $D_{ij}^{-1} \in n \times n$ $(j = 1, \ldots, l)$, with $\mu_{F_j}(s_i) = \exp\{-(s_i - m_{ij})^t D_{ij}^{-1}(s_i - m_{ij})\}$ (see Uemura, 1996), and $\mu_{F_j}(s_1, s_2) = \mu_{F_j}(s_1) \wedge \mu_{F_j}(s_2)$.

We obtain the fuzzy utility function in the following form:

$$\begin{aligned} &U_{F_J, D_i}(z) \\ &= ((c_{1i}^{-1} + c_{2i}^{-1})^{-1}(m_j - c_{1i}^{-1} - c_{2i}^{-1}), (c_{1i}^{-1} + c_{2i}^{-1})^{-1} D_{kj}^{-1})_e \end{aligned} \tag{13}$$

where c_{1i}^{-1}, c_{2i}^{-1} are the inverses of the previously introduced parameters, and we set the fuzzy utility function as equal $(mm_{ij}, DD_{ij}^{-1})_e$. Note that the fuzzy utility function is normal possibility distributed.

Then, we obtain the fuzzy expected utility function in the form

$$E(D_i) = \Big(\sum_j^m \Pi_{ij} mm_{ij}, \Big(\sum_j^m \Pi_{ij}^2 D_{ij} D_{ij}^{-1} \Big)^{-1} \Big)_e \tag{14}$$

where Π_{ij} is the possibility measure of a fuzzy event. Note that the fuzzy expected utility function is normal possibility distributed.

Therefrom, we obtain $EE(D_i)$ as follows:

$$EE(D_i) = \Big(\sum_j^m \Pi_{ij} w mm_{ij}, \Big(w \sum_k^m (\Pi_k^2 D_{ij} D_{ij}^{-1})^{-1} \Big) w^t \Big)_e \tag{15}$$

where $EE(D_i)$ is one-dimensionally normally distributed.

As the fuzzy numbers are normal possibility distributed, their ordering is based uniquely on the means of these fuzzy numbers, and thus we do not need the indexes for the ordering (see Uemura, 1996). Hence, we can deduce the following decision rule:

Decision Rule

We select as the optimal decision the one, for which there is

$$\max_i \sum_j^m \Pi_{ij} w mm_{ij}.$$

4. An example

A decision-making unit considers a plant and equipment investment. Production form the envisaged plant will be sold at high profits, but, on the other hand, this planned investment. The DM considers two potential decisions: D_1—making of investment, and D_2—reserved judgement. Here, the DM's "cool head" relates to the profit-oriented considerations, while her/his "warm heart"—to the wish of minimising pollution. The analysis of the problem is supported by the data concerning expected profit rates and the pollutant emissions, resulting from the DM's experience, associated also with the past investments. In the example we will assume that DM is risk neutral.

We can distinguish in our context two types of DMs based on relation between "warm heart" and "cool head". In the example here we consider a DM described as "warm heart" and "cool head".

First, we consider the decision problem no. 1, related to "cool head". The state of nature s_1 is assumed to correspond to the profit rate, and $\pi_1(s_1)$ is the prior distribution regarding the profit rate values. The DM constructs the utility functions $U_{(1,D_i)}(s_1)$ for $i = 1, 2$, via the certainty method of Keeney and Raiffa (1976). The DM defines for the decision D_1 two points: $(b_1, 1)$ and $(b_2, 0)$, and for the decision D_2: $(b_3, 0)$ and $(b_4, 1)$, the co-ordinates of these points corresponding to $(s_1, U_{(1,D_i)}(s_1))$, $i = 1, 2$. For this problem we obtain the following utility function, the appropriate conditions on b_i, $i = 1, \ldots, 4$, holding:

$$U_{(1,D_1)}(s_1) = \frac{1}{b_1 - b_2} s_1 - \frac{b_2}{b_1 - b_2} \tag{16}$$

$$U_{(1,D_2)}(s_1) = \frac{1}{b_4 - b_3} s_1 - \frac{b_3}{b_4 - b_3} \tag{17}$$

Then, the DM sets up the decision problem no. 2, related to "warm heart". Here, the state of nature, s_2, is the pollutant emission volume and $\pi_2(s_2)$ is the prior distribution of the pollutant emission. The respective utility functions $U_{(2,D_i)}(s_1)$ are constructed using the certainty method of Keeney and Raiffa (1976). For the decision D_1 the DM selects two points, $(a_1, 0)$ and $(a_2, 1)$, and for the decision D_2—the points $(a_3, 1)$ and $(a_4, 0)$, their co-ordinates corresponding to $(s_2, U_{(2,Di)}(s_2))$, $i = 1, 2$. The utility functions obtained therefrom are

$$U_{(2,D_1)}(s_2) = \frac{1}{a_2 - a_1} s_2 - \frac{a_1}{a_2 - a_1} \tag{18}$$

$$U_{(2,D_2)}(s_2) = \frac{1}{a_4 - a_3} s_2 - \frac{a_4}{a_4 - a_3} \tag{19}$$

Now, we set $U_{(1,Di)}(s_1) = n_i s_1 + l_i$ and $U_{(2,D_i)}(s_1) = nn_i s_1 + ll_i$, for $i = 1, 2$.

Third, the DM is facing only one fuzzy event F (the "astray" state), which has the possibility distribution $\mu_F(s_1, s_2)$, expressed through

$$\mu_F(s_1, s_2) = \exp \frac{\{s_1 + s_2 - (m_1 + m_2)\}^2}{c(b_1 + b_2)},$$

where m_1, m_2 are means and b_1, b_2 are the widths of the possibility distribution of combination for the two states of nature, decided upon by the DM on the basis of the common variance. In this case we obtain the following fuzzy utility functions

$$U_{D_1}(z) = \exp \frac{-\left[\frac{(n_1+n_2)}{n_1 n_2} z - \left(m_1 + m_2 - \frac{n_1 l_2 + n_2 l_1}{n_1 n_2}\right)\right]^2}{c(b_1+b_2)} \tag{20}$$

$$U_{D_2}(z) = \exp \frac{-\left[\frac{nn_1+nn_2}{nn_1 nn_2} z - \left(m_1 + m_2 - \frac{nn_1 ll_2 + nn_2 ll_1}{nn_1 nn_2}\right)\right]^2}{c(b_1+b_2)} \tag{21}$$

Now the DM calculates the fuzzy possibility measure for a fuzzy event Π_F from $\Pi_F = \max_{s1} \max_{s2}[\mu_F(s_1, s_2) \cdot \pi_1(s_1) \cdot \pi_2(s_2)]$. In this case we have the following expected fuzzy utility functions:

$$E_{D_1}(z) = \exp \frac{-\left[\frac{(n_1+n_2)}{n_1 n_2} z - \Pi_F\left(m_1 + m_2 - \frac{n_1 l_2 + n_2 l_1}{n_1 n_2}\right)\right]^2}{\Pi_F^2 c(b_1+b_2)} \tag{22}$$

$$E_{D_2}(z) = \exp \frac{-\left[\frac{nn_1+nn_2}{nn_1 nn_2} z - \Pi_F\left(m_1 + m_2 - \frac{nn_1 ll_2 + nn_2 ll_1}{nn_1 nn_2}\right)\right]^2}{\Pi_F^2 c(b_1+b_2)} \tag{23}$$

By the normal possibility decision rule of Dubois and Prade (1988) we obtain the ordering of the fuzzy expected utility function only on the basis of the means of the fuzzy expected utility function. Hence, as indicated before, we do not need to use an index for ordering of the fuzzy numbers. Thus, in this case the fuzzy decision rule is as follows:

fuzzy decision rule for the example

(1) When

$$\frac{m_1 + m_2 - \frac{n_1 l_2 + n_2 l_1}{n_1 n_2}}{\frac{n_1+n_2}{n_1 n_2}} \geq \frac{m_1 + m_2 - \frac{nn_1 ll_2 + nn_2 ll_1}{n_1 n_2}}{\frac{nn_1 + nn_2}{nn_1 nn_2}},$$

we choose D_1.

(2) When

$$\frac{m_1 + m_2 - \frac{n_1 l_2 + n_2 l_1}{n_1 n_2}}{\frac{n_1+n_2}{n_1 n_2}} \leq \frac{m_1 + m_2 - \frac{nn_1 ll_2 + nn_2 ll_1}{n_1 n_2}}{\frac{nn_1 + nn_2}{nn_1 nn_2}},$$

we choose D_2.

In this decision rule we can ignore Π_F, b_1, b_2, and c. This fuzzy decision means of the possibility function for the fuzzy events, while we use prior distributions and widths for one case only. In this example we assumed the DM to be risk neutral. Were s/he of some other type, we would have to obtain the fuzzy expected utility function, in accordance with Section 2, as a function that is normal possibility distribution.

5. Conclusion

The paper presents the way of combining two decision problems concerning a single (or a common) dimension, so that an effective fuzzy decision rule can be obtained. Normality of the possibility distribution is assumed, leading to possibility of fusing the respective functions related to the two decision problems and their characteristics (decisions, states of nature, utility functions, etc.). The approach proposed can be applied in cases when the statement of the problem requires making of more refined distinctions rather than considering simply a bi-criterion or bi-utility two-decision problem.

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