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ARITHMETIC SEQUENCES OF HIGHER DEGREES CHARACTERIZING FIGURATE NUMBERS

Grażyna Rygał[a], **Arkadiusz Bryll**[b], **Grzegorz Bryll**[c]

[a] *Pedagogical Faculty*
Jan Długosz University in Częstochowa
Armii Krajowej 13/15, 42-200 Częstochowa, Poland
e-mail: g.rygal@op.pl

[b] *Technical University of Częstochowa, The Faculty of Management*
Dąbrowskiego 69, 42-200 Częstochowa, Poland

[c] *Emeritus professor, Poland*

Abstract. Figurate numbers have simple geometric illustration: polygonal numbers can be represented by polygons, pyramidal numbers by pyramids, prismatic numbers by prisms, and trapezoidal numbers by trapezoids. The numbers mentioned above can be defined by formulae[1] or can be characterized by some arithmetic sequences of higher degrees which allow to obtain the corresponding formulae [3]. Figurate numbers due to their geometrical illustration and intersting properties can be of interest for school pupils.

1. Arithmetic sequences of higher degrees

For arbitrary sequence $\{a_n\} : a_1, a_2, a_3, \ldots$ we can calculate the sequence of the first finite differences $\{\Delta^1 a_n\}$

$$\Delta^1 a_1 = a_2 - a_1, \quad \Delta^1 a_2 = a_3 - a_2, \quad \Delta^1 a_3 = a_4 - a_3, \ldots \tag{1}$$

and the sequences of successive differences

$$\Delta^{k+1} a_i = \Delta^k a_{i+1} - \Delta^k a_i, \quad i = 1, 2, 3, \ldots \quad k = 1, 2, 3, \ldots \tag{2}$$

Using the method of complete mathematical induction it can be proved that an arbitrary term of a sequence $\{a_n\}$ can be described by the following formula (see [1, 4]):

[1] This method was used by W. Sierpiński in the book [5] in definitions of triangle and tetrahedral numbers.

$$a_n = \binom{n-1}{0} a_1 + \binom{n-1}{1} \Delta^1 a_1 + \binom{n-1}{2} \Delta^2 a_1 + \ldots + \binom{n-1}{n-1} \Delta^{n-1} a_1. \quad (3)$$

It is evident that to define the nth term of a sequence $\{a_n\}$ it is sufficient to (know) have the first term and the differences: $\Delta^1 a_1,\ \Delta^2 a_1,\ \ldots,\ \Delta^{n-1} a_1$.

A sequence $\{a_n\}$ is called *an arithmetic sequence of the degree* m ($m = 1, 2, \ldots$) if and only if the sequence $\{\Delta^m a_n\}$ is constant and differs from the zero sequence. A constant sequence will be called an arithmetic sequence of the zero degree.

It follows from Eq. (3) that an arbitrary term of an arithmetic sequence of the degree m is expressed as:

$$a_n = \binom{n-1}{0} a_1 + \binom{n-1}{1} \Delta^1 a_1 + \binom{n-1}{2} \Delta^2 a_1 + \ldots + \binom{n-1}{m} \Delta^m a_1. \quad (4)$$

To determine the differences $\Delta^1 a_1,\ \Delta^2 a_1, \ldots, \Delta^m a_1$ we draw up the following table

$$\begin{array}{lllll}
a_1 & & & & \\
 & \Delta^1 a_1 & & & \\
a_2 & & \Delta^2 a_1 & & \\
 & \Delta^1 a_2 & & \Delta^3 a_1 & \\
a_3 & & \Delta^2 a_2 & & \ddots \\
 & \Delta^1 a_3 & & \Delta^3 a_2 & \\
a_4 & & \Delta^2 a_3 & & \\
 & \Delta^1 a_4 & \vdots & \vdots & \\
a_5 & \vdots & \vdots & \vdots & \\
\vdots & \vdots & \vdots & \vdots &
\end{array} \quad (5)$$

2. Polygonal numbers

For sequences with general terms

(a) $a_n = n$,

(b) $a_n = 2n - 1$,

(c) $a_n = 3n - 2$,

(d) $a_n = 4n - 3$,

(e) $a_n = 5n - 4,$

$\ldots \ \ldots \ \ldots \ \ldots$

(f) $a_n = (s-2)n - (s-3), \qquad s = 3, 4, 5, \ldots$

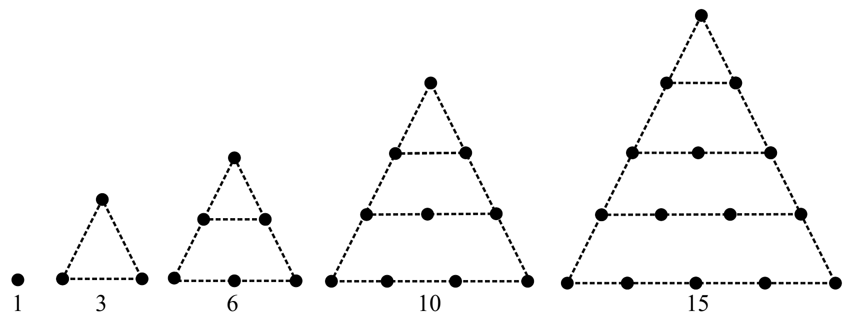
the sequences of partial sums[2] have the following form, respectively:

$\{t^{(3)}(n)\}:$	1	3	6	10	15	21	28	$\ldots$
$\{t^{(4)}(n)\}:$	1	4	9	16	25	36	49	$\ldots$
$\{t^{(5)}(n)\}:$	1	5	12	22	35	51	70	$\ldots$
$\{t^{(6)}(n)\}:$	1	6	15	28	45	66	91	$\ldots$
$\{t^{(7)}(n)\}:$	1	7	18	34	55	81	112	$\ldots$
$\ldots\ldots\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$
$\{t^{(s)}(n)\}:$	1	s	$3s-3$	$6s-8$	$10s-15$	$15s-24$	$21s-35$	$\ldots$

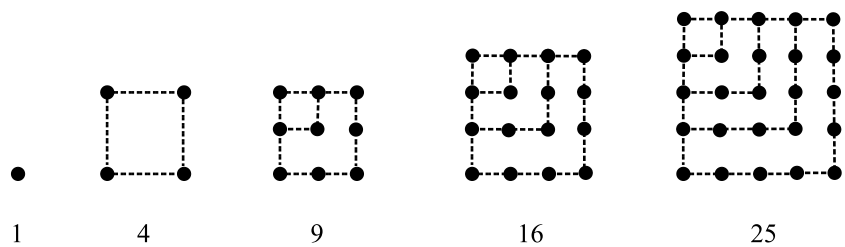
Elements (terms) of a sequence $\{t^{(s)}(n)\}$ are called s-gonal numbers. Hence, there are trigonal numbers, square (quaternal) numbers, pentagonal numbers, hexagonal numbers, etc.

Geometrical illustration of these numbers is as follows:

Trigonal numbers:

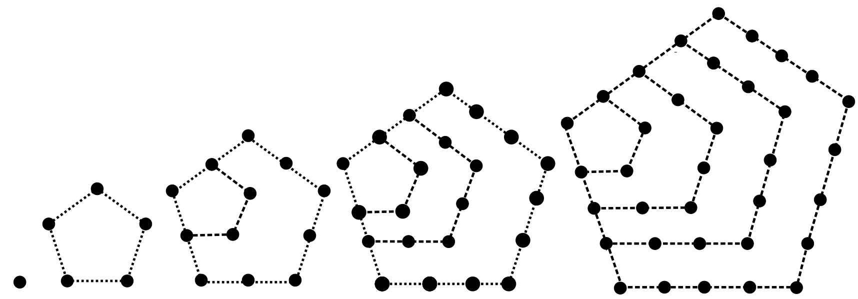


Square numbers:



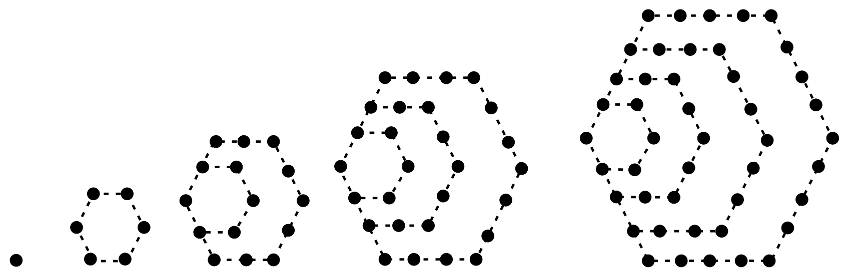
[2]The nth partial sum for a sequence $\{a_n\}$ is $a_1 + a_2 + \ldots + a_n$

Pentagonal numbers:



1 5 12 22 35

Hexagonal numbers:



1 6 15 28 45

The sequences $\{t^{(s)}(n)\}$ $(s = 3, 4, \dots)$ are arithmetic sequences of the second degree. The general term of a sequence $\{t^{(s)}(n)\}$ can be determined by the method of successive differences using table (5) and equation (4). The table of differences for this sequence has the form:

$$\begin{array}{cccc}
1 & & & \\
 & s-1 & & \\
s & & s-2 & \\
 & 2s-3 & & 0 \\
3s-3 & & s-2 & \\
 & 3s-5 & & 0 \\
6s-8 & & s-2 & \\
 & 4s-7 & \vdots & \vdots \\
10s-15 & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots
\end{array}$$

As $t^{(s)}(1) = 1$, $\Delta^1 t^{(s)}(1) = s-1$, $\Delta^2 t^{(s)}(1) = s-2$, then, using Eq. (4), we obtain:

$$t^{(s)}(n) = \binom{n-1}{0} \cdot 1 + \binom{n-1}{1}(s-1) + \binom{n-1}{2}(s-2),$$

i.e.

$$t^{(s)}(n) = \frac{n}{2}[n(s-2) - s + 4]. \tag{6}$$

Therefore trigonal, square, pentagonal, hexagonal, and heptagonal numbers can be defined using the following equations:

$$t^{(3)}(n) = \frac{n}{2}(n+1), \qquad t^{(4)}(n) = n^2,$$

$$t^{(5)}(n) = \frac{n}{2}(3n-1), \qquad t^{(6)}(n) = \frac{n}{2}(4n-2),$$

$$t^{(7)}(n) = \frac{n}{2}(5n-3).$$

Many properties of s-gonal numbers can be found in books [2, 5].

3. Pyramidal numbers

If for a sequence $\{t^{(s)}(n)\}$ the sequence of partial sums is created, then a sequence $\{T^{(s)}(n)\}$ is obtained being an arithmetic sequence of the third degree. Elements (terms) of this sequence are called *s-gonal pyramidal numbers.*

The sequences of trigonal, square, pentagonal, hexagonal, heptagonal, ..., s-gonal pyramidal numbers have the following form

$\{T^{(3)}(n)\}:$	1	4	10	20	35	...
$\{T^{(4)}(n)\}:$	1	5	14	30	55	...
$\{T^{(5)}(n)\}:$	1	6	18	40	75	...
$\{T^{(6)}(n)\}:$	1	7	22	50	95	...
$\{T^{(7)}(n)\}:$	1	8	26	60	105	...
......	...	...	...	...	...	...
$\{T^{(s)}(n)\}:$	1	$s+1$	$4s-2$	$10s-10$	$20s-25$	...

respectively.

To determine the general term of a sequence $\{T^{(s)}(n)\}$ draw up a table of successive differences:

$$
\begin{array}{ccccc}
1 & & & & \\
 & s & & & \\
s+1 & & 2s-3 & & \\
 & 3s-3 & & s-2 & \\
4s-2 & & 3s-5 & & 0 \\
 & 6s-8 & & s-2 & \\
10s-10 & & 4s-7 & \vdots & \vdots \\
 & 10s-15 & \vdots & \vdots & \vdots \\
20s-25 & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots
\end{array}
$$

For the sequences under consideration we have

$$T^{(s)}(1) = 1, \qquad \Delta^1 T^{(s)}(1) = s$$

$$\Delta^2 T^{(s)}(1) = 2s-3, \qquad \Delta^3 T^{(s)}(1) = s-2.$$

Using Eq. (4), we obtain

$$T^{(s)}(n) = \binom{n-1}{0} \cdot 1 + \binom{n-1}{1} s + \binom{n-1}{2} \cdot (2s-3) + \binom{n-1}{3} (s-2),$$

i.e.

$$T^{(s)}(n) = \frac{n}{6}[n^2(s-2) + 3n - (s-5)]. \tag{7}$$

This eqiuation can be written as

$$T^{(s)}(n) = \frac{n}{6}(n+1)[n(s-2) - (s-5)]. \tag{8}$$

Indeed:

$$n^2(s-2) + 3n - (s-5) = n^2(s-2) + 3n + (s-2) + 3 =$$

$$(n^2-1)(s-2) + 3(n+1) = (n+1)[(n-1)(s-2) + 3] =$$

$$= (n+1)[n(s-2) - (s-5)].$$

For sequences $\{T^{(3)}(n)\}, \ldots, \{T^{(7)}(n)\}$, according to Eq. (8), the general terms have the form:

$T^{(3)}(n) = \frac{n}{6}(n+1)(n+2), \qquad T^{(4)}(n) = \frac{n}{6}(n+1)(2n+1),$

$T^{(5)}(n) = \frac{n^2}{2}(n+1), \qquad T^{(6)}(n) = \frac{n}{6}(n+1)(4n-1),$
$T^{(7)}(n) = \frac{n}{6} \cdot (n+1)(5n-2).$

In the book [5] many properties of trigonal pyramidal numbers being elements (terms) of the sequence $T^{(3)}(n)$ can be found.

4. Prismatic numbers

Let m be an arbitrary natural number distinct from 1. Consider the sequence:

$$m, \quad 2(m+1), \quad 3(m+2), \quad 4(m+3), \quad 5(m+4), \ldots$$

and create for it the sequence $\{P^{(m)}(n)\}$ being the sequence of partial sums:

$$\{P^{(m)}(n)\}: \quad m, \quad 3m+2, \quad 6m+8, \quad 10m+20, \quad 15m+40, \ldots$$

It is easy to see that the above sequence is an arithmetic sequence of the third degree. The general term of this sequence cab be found drawing up a table of successive differences

$$\begin{array}{ccccc}
m & & & & \\
 & 2m+2 & & & \\
3m+2 & & m+4 & & \\
 & 3m+6 & & 2 & \\
6m+8 & & m+6 & & 0 \\
 & 4m+12 & & 2 & \\
10m+20 & & m+8 & & 0 \\
 & 5m+20 & & 2 & \vdots \\
15m+40 & & m+10 & & \vdots \\
 & 6m+30 & \vdots & \vdots & \vdots \\
21m+70 & \vdots & \vdots & \vdots & \\
\vdots & \vdots & \vdots & \vdots & \vdots
\end{array}$$

Then we have

$$P^{(m)}(1) = m, \qquad \Delta^1 P^{(m)}(1) = 2m+2,$$
$$\Delta^2 P^{(m)}(1) = m+4, \qquad \Delta^3 P^{(m)}(1) = 2.$$

Using Eq. (4), we obtain

$$P^{(m)}(n) = \binom{n-1}{0}m + \binom{n-1}{1}2(m+1) + \binom{n-1}{2}(m+4) + \binom{n-1}{3}\cdot 2$$

or

$$P^{(m)}(n) = \frac{n}{6}(n+1)[3m + 2(n-1)]. \tag{9}$$

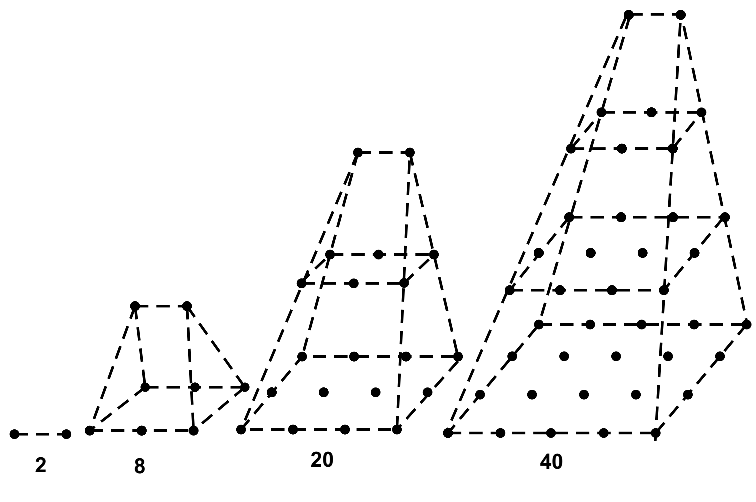
Elements (terms) of the sequence $\{P^{(m)}(n)\}$ are called *prismatic numbers of the range* m ($m \geq 2$.)

The sequences of prismatic numbers of the second, third, and forth range have the form:

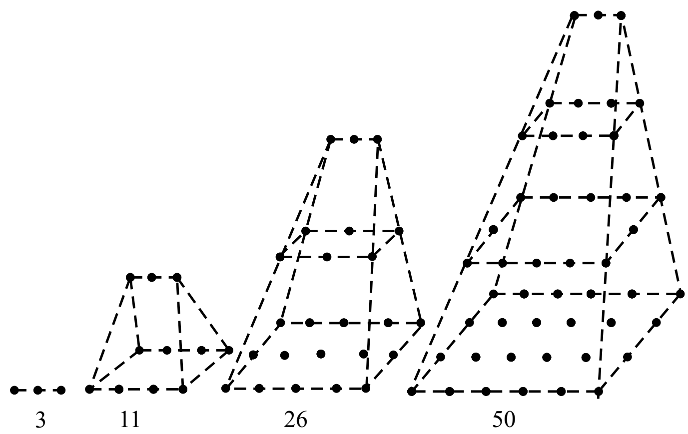
$\{P^{(2)}(n)\}$:	2	8	20	40	70	...	$P^{(2)}(n) = \frac{1}{3}(n+1)(n+2)$
$\{P^{(3)}(n)\}$:	3	11	26	50	85	...	$P^{(3)}(n) = \frac{n}{6}(n+1)(2n+7)$
$\{P^{(4)}(n)\}$:	4	16	32	60	100	...	$P^{(4)}(n) = \frac{n}{3}(n+1)(n+5)$

Using prism, the prismatic numbers have the following geometrical illustration:

$m = 2$



$m = 3$



It should be noted that faces of a prism illustrating prismatic numbers being triangles can be interpreted as corresponding trigoanl numbers, whereas lateral faces of a prism being trapezes as *trapezoid numbers*.

For numbers $m = 2,\ m = 3,\ m = 4, \ldots, m = s$ we obtain the following sequences of trapezoid numbers:

$\{R^{(2)}(n)\}:$	2	5	9	14	20	$\ldots$
$\{R^{(3)}(n)\}:$	3	7	12	18	25	$\ldots$
$\{R^{(4)}(n)\}:$	4	9	15	22	30	$\ldots$
....	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$
$\{R^{(s)}(n)\}:$	s	$2s+1$	$3s+3$	$4s+6$	$5s+10$	$\ldots$

The general term of the sequence $\{R^{(s)}(n)\}$ can be found based on a table of successive differences:

$$\begin{array}{cccc}
s & & & \\
 & s+1 & & \\
2s+1 & & 1 & \\
 & s+2 & & 0 \\
3s+3 & & 1 & \\
 & s+3 & & 0 \\
4s+6 & & 1 & \\
 & s+4 & \vdots & \vdots \\
5s+10 & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots
\end{array}$$

Hence, we have $R^{(s)}(1) = s$, $\Delta^1 R^{(s)}(1) = s + 1$, $\Delta^2 R^{(s)}(1) = 1$. Using Eq. (4), we obtain

$$R^{(s)}(n) = \binom{n-1}{0} s + \binom{n-1}{1}(s+1) + \binom{n-1}{2}$$

or

$$R^{(s)}(n) = \frac{n}{2}(n - 1 + 2s). \tag{10}$$

The sequence $\{R^{(s)}(n)\}$ is an arithmetic sequemce of the second degree.

According to (10), the sequences $\{R^{(2)}(n)\}$, $\{R^{(3)}(n)\}$, $\{R^{(4)}(n)\}$ have the following general terms:

$$R^{(2)}(n) = \frac{n}{2}(n+3), \qquad R^{(3)}(n) = \frac{n}{2}(n+5), \qquad R^{(4)}(n) = \frac{n}{2}(n+7).$$

Summarizing we state that all the types of the abovementioned figurate numbers can be characterized by some arithmetic sequences of the second or third degree.

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