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Roman WITUŁA, Danuta JAMA, Iwona NOWAK, Paweł OLCZYK
Institute of Mathematics
Silesian University of Technology

VARIATIONS ON SEQUENCES OF ARITHMETIC AND GEOMETRIC MEANS

Summary. The paper presents a thematic overview and selected results connected with the asymptotic behavior of sequences of arithmetic and geometric means of fixed sequences of positive real numbers. A lot of original results and the independent proofs of known results are presented. Some rarely cited classical results (including the Kalecki Theorem and the Hurwitz identity) are recalled and used.

WARIACJE NA TEMAT CIĄGÓW ŚREDNICH ARYTMETYCZNYCH I GEOMETRYCZNYCH

Streszczenie. W artykule przedstawiono przegląd tematyczny oraz wybrane wyniki dotyczące asymptotycznych zachowań ciągów średnich arytmetycznych i geometrycznych danych ciągów liczb dodatnich. Podano wiele orginalnych wyników oraz niezależnych dowodów znanych faktów. Przypomniano i zastosowano kilka, rzadko cytowanych wyników klasycznych (m.in. twierdzenie Kaleckiego, tożsamość Hurwitza).

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Paweł Olczyk was a student of the Faculty of Mathematics and Physics in Silesian University of Technology. At present he is employed in Dynamic Technologies Polska.

1. Introduction

Our work, being a review and complementary, concerns a discussion of the asymptotic dependence for arithmetic and geometric means of fixed sequences of positive real numbers. In this work we identified some common areas of research. In the first chapter we present the application in number theory, related to a sequence of consecutive primes. An important topic here seems to be the almost forgotten Kalecki's Theorem, which concerns asymptoticity of certain sums including sums of positive powers of consecutive primes. In the second chapter we study the sequence $\{G_n(a_i)/a_n\}$, where $\{a_n\}$ is a sequence of positive numbers. We estimate lower and upper limits of this sequence, we discuss the convergence of this one. In the third chapter we give asymptoticity of sequences $\{G_n(\ln^\alpha(i)+\varepsilon_i)\}$ and $\{A_n(\ln^\alpha(i)+\varepsilon_i)\}$, where $\alpha > 0$, $\{\varepsilon_i\} \subset [0,\infty)$ is a sequence of disturbances. Finally, in the fourth chapter we give the Hurwitz identity, which because of its character can be used to estimate the difference between $A_n(a_i)$ and $G_n(a_i)$ for some sequences $\{a_i\}$ of positive real numbers.

It should be noted that particularly important topics discussed here are works of Jakimczuk [2–5], who introduced the so-called slow increase functions.

In this paper we assume that $A_n(a_i)$ ($G_n(a_i)$, resp.) denotes the arithmetical mean (geometrical mean resp.) of the first n-elements of the given sequence $\{a_i\}_{i=1}^{\infty} \subset [0,\infty)$ for every $n \in \mathbb{N}$. The natural logarithm will be denoted by $\log(\cdot)$.

2. Sequence of consecutive prime numbers

The result below is known [3], but presented proof seems to be original.

Theorem 1. *Let $\{p_i\}_{i=1}^{\infty}$ be the increasing sequence of all prime numbers. Then we get*

$$\lim_{n\to\infty} A_n(p_i)\,(n\log(n))^{-1} = \frac{1}{2}$$

and

$$\lim_{n\to\infty} G_n(p_i)\,(n\log(n))^{-1} = \frac{1}{e}$$

Proof. At the beginning we show that the following auxiliary relations are satisfied:

$$\sum_{i=2}^{n} i\log(i) = \frac{1}{2}n^2\log(n) + O(n^2) \tag{1}$$

and

$$\sum_{i=2}^{n} i \log\log(i) = \frac{1}{2} n^2 \log\log(n) + O\left(\frac{n^2}{\log(n)}\right). \tag{2}$$

For every $i \in \mathbb{N}$ we get

$$\begin{aligned}
(i+1)^2 \log(i+1) - i^2 \log(i) &= \\
= i^2 \log\left(1 + \frac{1}{i}\right) + (2i+1)\log(i+1) &= \\
= 2i\log(i) + O(i)&,
\end{aligned}$$

thus after adding up sides for $i = 2, 3, \ldots, n-1$, we obtain

$$n^2 \log(n) = 2\sum_{i=2}^{n} i\log(i) + O(n^2),$$

which implies the relation (1).

Now, for every $i \in \mathbb{N}$ we deduce that

$$\begin{aligned}
(i+1)^2 \log\log(i+1) - i^2 \log\log(i) &= \\
= i^2 \log\left(1 + \frac{\log\left(1+\frac{1}{i}\right)}{\log(i)}\right) + (2i+1)\log\log(i+1) &= \\
= 2i\log\log(i) + O\left(\frac{i}{\log(i)}\right)&.
\end{aligned}$$

Thus after adding up sides for $i = 2, 3, \ldots, n-1$, we obtain

$$n^2 \log\log(n) = 2\sum_{i=2}^{n} i\log(i) + O\left(\frac{n^2}{\log(n)}\right),$$

where the relation below was used:

$$\sum_{i=2}^{n} \frac{i}{\log(i)} = O\left(\frac{n^2}{\log(n)}\right),$$

which, in turn, easily follows from the identity:

$$\begin{aligned}
\frac{(n+1)^2}{\log(n+1)} - \frac{n^2}{\log(n)} &= \\
= \frac{2(n+1)}{\log(n+1)} - \frac{1}{\log(n+1)} - n^2 \frac{\log\left(1+\frac{1}{n}\right)}{\log(n)\log(n+1)}&
\end{aligned}$$

$$= \frac{2(n+1)}{\log(n+1)}\left(1 - \frac{1}{2(n+1)} - \frac{n^2 \log\left(1+\frac{1}{n}\right)}{2(n+1)\log(n)}\right)$$

and the fact that if $\{a_n\} \subset \mathbb{R}_+$, $\{b_n\} \subset \mathbb{R}$,

$$\sum_{n=1}^{\infty} a_n = \infty \quad \text{and} \quad \lim_{n\to\infty} b_n = 0, \quad \text{then} \quad \sum_{k=1}^{n} b_k a_k = o\left(\sum_{k=1}^{n} a_k\right).$$

Now, we are ready to work on direct proof of relation in the conclusion of the theorem. Using (1), (2) and the following Rosser–Schoenfeld's inequalities (see [8], point 5.26 or source work of Rosser and Schoenfeld [14]):

$$n\left(\log(n) + \log\log(n) - \frac{3}{2}\right) < p_n < n\left(\log(n) + \log\log(n) - \frac{1}{2}\right) \tag{3}$$

which are satisfied for any $n \in \mathbb{N}$, $n \geqslant 20$, we obtain

$$A_n(p_i) = \frac{1}{2} n\log(n) + O(n \log\log(n))$$

and in consequence

$$\lim_{n\to\infty} A_n(p_i)(n\log(n))^{-1} = \frac{1}{2}.$$

In turn,

$$G_n(p_i) = G_n(i) G_n(\log(i)) G_n\left(1 + \frac{\log\log(i) - a(i)}{\log(i)}\right),$$

where $a(i) \in \left(-\frac{3}{2}, -\frac{1}{2}\right)$ for $i \geqslant 20$. Without lost of generality of the discussion, we can assume that $a(i) := \log\log(i)$, $\log\log(i) := 1$ and $\log(i) := 1$ for all $i \leqslant \left\lfloor e^{e^{\frac{3}{2}}} \right\rfloor$. Of course $G_n(i) = \sqrt[n]{n!}$. On the other hand, from the proof of given below Theorem 14 results that

$$G_n(\log(i)) = \log(n) - 1 + O\left(\frac{1}{\log(n)}\right).$$

Finally

$$\begin{aligned}
1 &\leqslant G_n\left(1 + \frac{\log\log(i) - a(i)}{\log(i)}\right) \leqslant \\
&\leqslant \exp\left(\frac{1}{n}\sum_{i=2}^{n} \frac{\log\log(i) - a(i)}{\log(i)}\right) \leqslant \\
&\leqslant \exp\left(\frac{\log\log(n)}{n}\sum_{i=2}^{n}\frac{1}{\log(i)}\right) = \\
&= \exp\left(\frac{\log\log(n)}{\log(n)} + o\left(\frac{\log\log(n)}{\log(n)}\right)\right) \xrightarrow{n\to\infty} 1.
\end{aligned}$$

Summarizing, we obtain

$$G_n(p_i) = \left(\frac{1}{n}\sqrt[n]{n!}\right)(n\log(n))\alpha_n$$

where $\alpha_n := \left(1 + O\left(\frac{1}{\log(n)}\right)\right) G_n\left(1 + \frac{\log\log(i) - a(i)}{\log i}\right)$ tends to 1 while n tends to infinity. $\square$

Remark 2. From inequality (3) one can deduces the following relation (see [2,15]):

$$A_n(p_i^\alpha) \approx \frac{p_n^\alpha}{\alpha + 1} \approx \frac{n^\alpha \log^\alpha n}{\alpha + 1}, \quad \text{where} \quad \alpha > 0. \tag{4}$$

Remark 3. The sequences $A_n(p_i^\alpha)$ and $G_n(p_i^\alpha)$ are connected with more general sequences:

$$A_n(f(p_i)) \quad \text{and} \quad G_n(f(p_i)),$$

where $f : (0, \infty) \to (0, \infty)$ is nondecreasing function satisfying the condition: for each $u > 0$ there exists the limit:

$$\lim_{x\to\infty} \frac{f(ux)}{f(x)} = \varphi(u).$$

As it has been proved by Michał Kalecki [6] (nomen omen one of the most outstanding polish economists – in 1970 he aspired to The Noble Prize in economy), then there exists $s \geqslant 0$, such that:

$$\varphi(u) = u^s \quad \text{and} \quad nA_n(f(p_i)) = \frac{1 + o(1)}{s + 1} f(p_n) \frac{p_n}{\log p_n} \overset{(3)}{=} \frac{1 + o(1)}{s + 1} f(p_n) n,$$

i.e.

$$A_n(f(p_i)) = \frac{1 + o(1)}{s + 1} f(p_n). \tag{5}$$

Kalecki Theorem is a significant generalization of early E. Landau's result and was directly derived from Hadamard – de la Vallée-Poussin Prime Number Theorem: $\pi(N) = \frac{N}{\log N}(1 + o(1))$, where $\pi(N)$ is a number of prime numbers $p \leqslant N$.

Let us notice, that from (5) arises (4) and the first equation of the Theorem 1

Furthermore, if for the function f discussed here we additionally assume that $f(x) > 1$ for $x > 0$ and $\lim_{x\to\infty} \ln f(x) = \infty$, then we obtain

$$s^* := \frac{\ln f(ux)}{\ln f(x)} = 1 + \frac{\ln \frac{f(ux)}{f(x)}}{\ln f(x)} \xrightarrow{x\to\infty} 1$$

and

$$\ln G_n(f(p_i)) = A_n(\ln f(p_i)) \overset{(5)}{=} (1 + o(1)) \ln f(p_n).$$

3. Properties of the sequence $\{a_n^{-1}G_n(a_i)\}$

Let us start with the fundamental technical result of this section:

Theorem 4. *Let* $\{a_n\} \subset \mathbb{R}_+$ *be a nondecreasing sequence,* $\lim\limits_{n\to\infty} a_n = \infty$.

1. *If* $\limsup\limits_{n\to\infty} n(a_{n+1} - a_n) < \infty$*, then* $\lim\limits_{n\to\infty} a_n^{-1}G_n(a_i) = 1$.
2. *If* $\limsup\limits_{n\to\infty} \left(\sum\limits_{i=1}^{n-1} \frac{i}{n}(1 - a_i a_{i+1}^{-1})\right) = +\infty$*, then* $\lim\limits_{n\to\infty} a_n^{-1}G_n(a_i) = 0$.
3. *The following general relations hold:*
$$\exp\left(\liminf_{n\to\infty}\left(\sum_{i=1}^{n-1}\frac{i}{n}(1 - a_{i+1}a_i^{-1})\right)\right) \leqslant \liminf_{n\to\infty} a_n^{-1}G_n(a_i)$$
and
$$\limsup_{n\to\infty} a_n^{-1}G_n(a_i) \leqslant \exp\left(\limsup_{n\to\infty}\left(\sum_{i=1}^{n-1}\frac{i}{n}(a_i a_{i+1}^{-1} - 1)\right)\right).$$

Proof. We have $\log(G(a_i)) = A_n(\log(a_i))$. Next
$$\sum_{i=1}^{n}\log(a_i) = n\log(a_1) + (n-1)\big(\log(a_2) - \log(a_1)\big) +$$
$$+(n-2)\big(\log(a_3) - \log(a_2)\big) + \ldots + \big(\log(a_n) - \log(a_{n-1})\big) =$$
$$= n\Big(\log(a_1) + \big(\log(a_2) - \log(a_1)\big) + \ldots + \big(\log(a_n) - \log(a_{n-1})\big)\Big) -$$
$$-\sum_{i=1}^{n-1} i\big(\log(a_{i+1}) - \log(a_i)\big) = n\log(a_n) - \sum_{i=1}^{n-1} i\big(\log(a_{i+1}) - \log(a_i)\big).$$
Hence by Mean Value Theorem we get
$$A_n(\log(a_i)) = \log(a_n) - \sum_{i=1}^{n-1}\frac{i}{n\xi_i}(a_{i+1} - a_i),$$
for some $\xi_i \in (a_i, a_{i+1})$ for every $i = 1, \ldots .n-1$.

Finally we obtain
$$a_n^{-1}G_n(a_i) = a_n^{-1}\exp\big(A_n(\log(a_i))\big) = \exp\left(\sum_{i=1}^{n-1}\frac{i}{n\xi_i}(a_i - a_{i+1})\right).$$

All three statements 1–3 of the Theorem 4, could be obtained from above relation in an elementary discussion. □

Corollary 5. *Let*

$$\begin{aligned} \log^{(1)}(x) &= \log(x), & x > 0, \\ \log^{(n+1)}(x) &= \log(\log^{(n)}(x)), & n \in \mathbb{N}. \end{aligned}$$

Then from statement 1 of the Theorem 4 we get

$$\lim_{n\to\infty} \left(\log^{(k)}(n)\right)^{-1} G_n\left(\log^{(k)}(i)\right) = 1,$$

for every $k \in \mathbb{N}$.

Corollary 6. *If we assume* $a_i = i$, *then from the statement 3 arises that*

$$\lim_{n\to\infty} \frac{\sqrt[n]{n!}}{n} = e^{-1}. \tag{6}$$

The refining of inequality (6) could be obtained by applying the Stirling's formula [13], which is anything unexpected but really robust technical instrument:

$$n! \sim \left(\frac{n}{e}\right)^n \sqrt{2\pi n}\left(1 + \frac{1}{12n} + \frac{1}{288n^2} - \frac{139}{51840n^3} + \dots\right). \tag{7}$$

Hence we obtain:

$$\left(e\frac{\sqrt[n]{n!}}{n}\right)^n \sim \sqrt{2\pi n}\left(1 + \frac{1}{12n} + \frac{1}{288n^2} + \dots\right),$$

which implies, the following limits (giving the unexpected relations between numbers π and e):

$$\lim_{n\to\infty} \frac{\left(e\frac{\sqrt[n]{n!}}{n}\right)^n}{\sqrt{2\pi n}} = 1,$$

$$\lim_{n\to\infty} \left(\frac{\left(e\frac{\sqrt[n]{n!}}{n}\right)^n}{\sqrt{2\pi n}}\right)^n = e^{\frac{1}{12}},$$

$$\lim_{n\to\infty} \left(e^{-\frac{1}{12}}\frac{\left(e\frac{\sqrt[n]{n!}}{n}\right)^n}{\sqrt{2\pi n}}\right)^n = 1.$$

We have used here, the auxiliary limit

$$\lim_{x\to\infty} \left(\frac{\left(1 + \frac{\alpha}{x} + \frac{\beta}{x^2} + \dots\right)^x}{e^\alpha}\right)^x = e^{\beta - \frac{\alpha^2}{2}}$$

which holds for any $\alpha, \beta \in \mathbb{R}$.

Romanian mathematician C. Mortici in work [9] has found the following asymptotic relation:

$$n! \sim \sqrt{2\pi} e^{-n-\frac{1}{2}} \left(n^2 + n + \frac{1}{6}\right)^{\frac{1}{2}n+\frac{1}{4}} = \sqrt{2\pi}\left(\frac{n}{e}\right)^{n+\frac{1}{2}} \left(1 + \frac{1}{n} + \frac{1}{6n^2}\right)^{\frac{1}{2}n+\frac{1}{4}}, \quad (8)$$

which was then generalized by R.B. Paris [11]. Because we have

$$\begin{aligned} \left(1 + \frac{1}{n} + \frac{1}{6n^2}\right)^{\frac{1}{2}n+\frac{1}{4}} &= \exp\left[\left(\frac{n}{2} + \frac{1}{4}\right) \log\left(1 + \frac{1}{n} + \frac{1}{6n^2}\right)\right] \\ &= \exp\left[\frac{1}{2} + \frac{1}{12n} - \frac{1}{144n^3} + O\left(\frac{1}{n^4}\right)\right] \\ &= e^{\frac{1}{2}}\left(1 + \frac{1}{12n} + \frac{1}{288n^2} - \frac{71}{10368n^3} + O\left(\frac{1}{n^4}\right)\right), \end{aligned}$$

so from formulae (7) and (8) we get

$$\lim_{n\to\infty} \left(\frac{n!}{\sqrt{2\pi}} \left(\frac{e}{\sqrt{n^2+n+\frac{1}{6}}}\right)^{n+\frac{1}{2}}\right)^{n^3} = \exp\left(\frac{71}{10368} - \frac{139}{51840}\right) = \exp\left(\frac{1}{240}\right).$$

Corollary 7. *If $\{a_n\} \subset \mathbb{R}_+$ is a nondecreasing sequence which enough fast diverges to ∞ so that $\limsup \frac{a_i}{a_{i+1}} < 1$, then as it results from the statement 2, the following relation holds*

$$\lim_{n\to\infty} a_n^{-1} G_n(a_i) = 0.$$

It is possible to accept the definitely weaker conditions for the elements a_i, $i \in \mathbb{N}$, for example that there exists $p \in (0,1)$, such that $\frac{a_i}{a_{i+1}} \leqslant 1 - i^{-p}$ for sufficiently large $i \in \mathbb{N}$.

The next lemma belongs to so-called mathematical folklore (see [7]):

Lemma 8. *If $a_i, b_i \in \mathbb{R}_+$, $1 \leqslant i \leqslant n$, then the following inequality exists:*

$$\sqrt[n]{(a_1+b_1)\cdot \ldots \cdot (a_n+b_n)} \geqslant \sqrt[n]{a_1 \cdot \ldots \cdot a_n} + \sqrt[n]{b_1 \cdot \ldots \cdot b_n}.$$

Remark 9. The above inequality can be proved by induction in analogical way as it was done by A. Cauchy for A-G inequality (so-called Cauchy binary induction, see [12]).

Corollary 10. *(see [1]) For any a_i, $c \in \mathbb{R}_+$, $i \in \mathbb{N}$, the following inequality holds*

$$G_n(a_i + c) \geqslant c + G_n(a_i)$$

for every $n \in \mathbb{N}$.

Corollary 11. *Let* $\{a_i\}_{i=1}^{\infty} \subset \mathbb{R}_+$*. If there exist* $r \geqslant 1$ *such that* $a_i \geqslant i^r$*,* $i = 1, 2, \ldots$*, then we have*

$$G_n(a_i) \geqslant \big(\sqrt[n]{n!}\big)^r + G_r\big(a_i - i^r\big),$$

which implies

$$\frac{G_n(a_i) - G_n(a_n - i^r)}{n^r} \geqslant \left(\frac{\sqrt[n]{n!}}{n}\right)^r \overset{by (6)}{\approx} e^{-r}.$$

Theorem 12. *Let* $\{a_i\} \subset \mathbb{R}$*,* $a_i \geqslant i$*,* $\varepsilon_i := a_i - i$*,* $i \in \mathbb{N}$*. Then the following optimal inequality is true*

$$(*) \qquad \limsup_{n\to\infty} \frac{1}{n}\left[A_n(a_i) - G_n(a_i)\right] \leqslant \frac{e-2}{2e} + \limsup_{n\to\infty} \frac{1}{n}\left[A_n(\varepsilon_i) - G_n(\varepsilon_i)\right].$$

Furthermore if $\lim_{i\to\infty} \varepsilon_i i^{-1} = 0$*, then* $\lim_{n\to\infty} n^{-1} G_n(a_i) = e^{-1}$*.*

If there exists $\delta \in \mathbb{R}_+$ *such that* $\varepsilon_i \leqslant \delta i$ *for all* $i \in \mathbb{N}$*, then*

$$\liminf_{n\to\infty} \frac{1}{n}\left[A_n(a_i) - G_n(a_i) - A_n(\varepsilon_i)\right] \geqslant \frac{1}{2} - (1+\delta)\frac{1}{e}.$$

Proof. We have

$$A_n(a_i) = \frac{n+1}{2} + A_n(\varepsilon_i),$$

then because of Corollary 11 we get

$$\frac{1}{n}\left[A_n(a_i) - G_n(a_i)\right] \leqslant \frac{n+1}{2n} - \frac{1}{n}\sqrt[n]{n!} - \frac{1}{n}\left[A_n(\varepsilon_i) - G_n(\varepsilon_i)\right],$$

for all $n \in \mathbb{N}$. The only thing remaining to prove $(*)$ is to notice that by (6) we have

$$\lim_{n\to\infty}\left(\frac{n+1}{2n} - \frac{1}{n}\sqrt[n]{n!}\right) = \frac{e-2}{2e}.$$

Let us assume now that $\lim_{i\to\infty} \varepsilon_i i^{-1} = 0$. Then

$$G_n(a_i) = \sqrt[n]{n!}\left[\prod_{i=1}^{n}\big(1 + \varepsilon_i i^{-1}\big)\right]^{\frac{1}{n}}.$$

Of course we have

$$\lim_{n\to\infty}\left[\prod_{i=1}^{n}\big(1 + \varepsilon_i i^{-1}\big)\right]^{\frac{1}{n}} = 1$$

and the conclusion arises from (6). Let finally suppose that there exists $\delta \in [0, \frac{e-2}{2}]$ such that $\varepsilon_i \leqslant \delta i$ for all $i \in \mathbb{N}$. Then the difference between arithmetic and geometric means is estimated in the following way:

$$A_n(a_i) - G_n(a_i) = \frac{n+1}{2} + A_n(\varepsilon_i) - \sqrt[n]{n!}\left[\prod_{i=1}^{n}\left(1+\varepsilon_i i^{-1}\right)\right]^{\frac{1}{n}} \geqslant$$

$$\geqslant \frac{n+1}{2} + A_n(\varepsilon_i) - \frac{1}{n}\sqrt[n]{n!}\left(n + \sum_{i=1}^{n}\varepsilon_i i^{-1}\right) \geqslant$$

$$\geqslant \left(\frac{1}{2} - (1+\delta)\frac{1}{n}\sqrt[n]{n!}\right)n + \frac{1}{2} + A_n(\varepsilon_i).$$

The final estimation from (6) follows. Additionally for this moment the information is the sequence $\left\{\frac{1}{n}\sqrt[n]{n!}\right\}$ is decreasing (the strenthening of the relation (6) – for the proof see the identity (9) below). □

Remark 13. We note that

$$\frac{n+1}{\sqrt[n+1]{(n+1)!}} = \left(\prod_{k=1}^{n}\left(1+\frac{1}{k}\right)^{k}\right)^{\frac{1}{n+1}} \tag{9}$$

and (see [12]):

$$\frac{e}{2n+2} < e - \left(1+\frac{1}{n}\right)^{n} < \frac{e}{2n+1}.$$

Hence, we deduce the estimation

$$\frac{n+1}{\sqrt[n+1]{(n+1)!}} < e^{\frac{n}{n+1}} < e\left(1 - \frac{1}{n+1} + \frac{1}{2(n+1)^2}\right),$$

i.e.,

$$\frac{\sqrt[n+1]{(n+1)!}}{n+1} - \frac{1}{e} > \frac{\frac{1}{n+1} - \frac{1}{2(n+1)^2}}{e\left(1 - \frac{1}{n+1} + \frac{1}{2(n+1)^2}\right)} = \frac{1 - \frac{1}{2(n+1)}}{e\left(n + \frac{1}{2(n+1)}\right)}.$$

On the other hand we have

$$\frac{n+1}{\sqrt[n+1]{(n+1)!}} > \left(\frac{\prod_{k=1}^{n}\left(1+\frac{1}{k}\right)^{k+1}}{\prod_{k=1}^{n}\left(1+\frac{1}{k}\right)}\right)^{\frac{1}{n+1}} > \frac{e^{\frac{n}{n+1}}}{\sqrt[n+1]{n+1}} > e\left(\frac{e}{n+1}\right)^{\frac{1}{n+1}},$$

which implies

$$\frac{\sqrt[n+1]{(n+1)!}}{n+1} < \frac{1}{e}\left(\frac{n+1}{e}\right)^{\frac{1}{n+1}} < \frac{1}{e}\left(1 + 2\frac{\log(n+1)-1}{n+1}\right)$$

if $\frac{\log(n+1)-1}{n+1} < \log 2$ since $\left(\frac{n+1}{e}\right)^{\frac{1}{n+1}} = \exp\left(\frac{\log(n+1)-1}{n+1}\right)$ and $1+2x > e^x$ for $x \in (0, \log 2)$. At last we get

$$\frac{\sqrt[n+1]{(n+1)!}}{n+1} - \frac{1}{e} < \frac{2}{e}\frac{\log(n+1)-1}{n+1}.$$

In the above considerations the following known inequalities were applied:

$$\Big(1+\frac{1}{n}\Big)^n < e < \Big(1+\frac{1}{n}\Big)^{n+1}, \qquad n \in \mathbb{N}.$$

4. Logarithmic sequences

The next theorem provides an asymptoticy of a difference

$$A_n(\log^a(i)) - G_n(\log^a(i)),$$

where $a > 0$ and $\log^a(1) := 1$. The sequence $a_i = \log^a(i)$, $i = 1, 2, \dots$ is a fundamental example of sequence which we meet in practice and for which we have „similar" asymptoticity of sequences $A_n(a_i)$ and $G_n(a_i)$. It is possible to generate this kind of relations also for different so called functions of slow increase (see [2, 4]). After Jakimczuk the function $f : [a, \infty) \to (0, \infty)$ is said to be of slow increase if the following condition holds:

$$\lim_{x\to\infty} \frac{f'(x)}{\frac{f(x)}{x}} = 0.$$

Examples of functions of slow increase are $f(x) = \log x$, $f(x) = \log^2 x$, $f(x) = \log\log x$, $f(x) = \frac{\log x}{\log\log x}$ and the psi function $\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}$ where $\Gamma(x)$ is the gamma function (see [13]).

Theorem 14. *For each $a \in \mathbb{R}_+$ the following relation holds:*

$$\Big(A_n\big(\log^a(i)\big) - G_n\big(\log^a(i)\big)\Big)\Big(\log(n)\Big)^{2-a} = \\ = \frac{1}{2}a^2 + \frac{1}{6}a^2(12-5a)\log^{-1}(n) + O\Big(\log^{-2}(n)\Big). \quad (10)$$

Proof. Let $a \in (1, +\infty)$. Then we get (see [10] Th.2.4)

$$A_n\big(\log^a(i)\big) = \frac{1}{n}\int_e^n \log^a(x)dx + O\left(\frac{1}{n}\log^a(n)\right) =$$
$$= \log^a(n) - \frac{a}{n}\int_e^n \log^{a-1}(x)dx + O\left(\frac{1}{n}\log^a(n)\right) =$$

(after integrating by parts)

$$= \log^a(n) - a\log^{a-1}(n) - a(a-1)\log^{a-2}(n) -$$
$$- a(a-1)(a-2)\log^{a-3}(n) + O\left(\log^{a-4}(n)\right)$$

and

$$G_n\big(\log^a(i)\big) = \exp\Big(A_n\big(\log(\log^a(i))\big)\Big) =$$
$$= \exp\left(\frac{a}{n}\int_e^a \log\log(x)dx + O\left(\frac{1}{n}\log\log(n)\right)\right) =$$
$$= \exp\left(\log(\log^a(n)) - \frac{a}{n}\int_e^a \log^{-1}(x)dx + O\left(\frac{1}{n}\log\log(n)\right)\right) =$$
$$= \log^a(n)\exp\left(-a\log^{-1}(n) - a\log^{-2}(n) - 2a\log^{-3}(n) + O\left(\log^{-4}(n)\right)\right) =$$
$$= \log^a(n) - a\log^{a-1}(n) + \left(\frac{1}{2}a^2 - a\right)\log^{a-2}(n) +$$
$$+ \left(a^2 - 2a - \frac{1}{6}a^3\right)\log^{a-3}(n) + O\left(\log^{a-4}(n)\right).$$

It results in

$$A_n\big(\log^a(i)\big) - G_n\big(\log^a(i)\big) =$$
$$= \frac{1}{2}a^2\log^{a-2}(n) + \frac{1}{6}a^2(12 - 5a)\log^{a-3}(n) + O\left(\log^{a-4}(n)\right),$$

what implies the expected relation. □

As a supplement of above theorem, we will present the result concerning the asymtotic behavior of differences between A-G means disturbed logarithmic sequences.

Theorem 15. *Let* $\{\varepsilon_i\}_{i=2}^{\infty} \subset [0,\infty)$. *If* $\sum\limits_{i=2}^{\infty} \varepsilon_i \log^{-a}(i) < \infty$, *then*

$$A_n\big(\log^a(i)+\varepsilon_i\big) - G_n\big(\log^a(i)+\varepsilon_i\big) = \\ = A_n\big(\log^a(i)\big) - G_n\big(\log^a(i)\big) + O\left(n^{-1}\log^a(n)\right). \quad (11)$$

Proof. It is easy to show that

$$A_n\big(\log^a(i)+\varepsilon_i\big) - G_n\big(\log^a(i)+\varepsilon_i\big) = A_n\big(\log^a(i)\big) - G_n\big(\log^a(i)\big) + \\ + A_n(\varepsilon_i) - G_n(\log^a(i)) \cdot (1 - G_n(1+\varepsilon_i \log^{-a}(i))$$

with

$$A_n(\varepsilon_i) = O\big(n^{-1}\log^a(n)\big),$$

Indeed, if for any $M>0$ there exists $n \in \mathbb{N}$, such that

$$\varepsilon_2 + \ldots + \varepsilon_{n+1} > M \log^a(n),$$

which means

$$\sum_{i=2}^{n+1} \varepsilon_i \log^{-a}(i) \geqslant \sum_{i=2}^{n+1} \varepsilon_i \log^{-a}(n) > M,$$

which implies the contradiction with assumed convergence of the series $\sum\limits_{i=2}^{\infty} \varepsilon_i \log^{-a}(i)$. On the other hand from the proof of Theorem 14 arises

$$G_n(\log^a(i)) = \log^a(i) + O\big(\log^{a-1}(n)\big).$$

It remains to note that

$$0 < G_n(1+\varepsilon_i \log^{-a}(i)) - 1 = \exp\left(\frac{1}{n}\sum_{i=2}^{n+1} \log(1+\varepsilon_i \log^{-a}(i))\right) - 1 < \\ < \exp\left(\frac{1}{n}\sum_{i=2}^{n+1} \varepsilon_i \log^{-a}(i)\right) - 1 < \exp\left(\frac{1}{n}\sum_{i=2}^{\infty} \varepsilon_i \log^{-a}(i)\right) - 1 = O(n^{-1}).$$

□

Remark 16. (see [16, 17]) If in Theorem 15 we assume weak assumption that $\{\varepsilon_i\}_{i=2}^{\infty} \subset \mathbb{R}$ and that series $\sum\limits_{i=2}^{\infty} \varepsilon_i \log^{-a}(i)$, $\sum\limits_{i=2}^{\infty} \varepsilon_i^2 \log^{-2a}(i)$ are both convergent with $\varepsilon_i \log^{-a}(i) > -1$ and $\log^a(i) + \varepsilon_i \geqslant 0$ for all $i \in \mathbb{N}$, then the equation (11) takes the form:

$$A_n\big(\log^a(i)+\varepsilon_i\big) - G_n\big(\log^a(i)+\varepsilon_i\big) = \\ = A_n\big(\log^a(i)\big) + A_n(\varepsilon_i) - G_n\big(\log^a(i)\big) + O\left(n^{-1}\right).$$

for all $n \in \mathbb{N}$. If we omit the last condition then the equation is still satisfied for every odd positive integer n. Then it should be notice that

$$G_n\left(1+\varepsilon_i \log^{-a}(i)\right) = \exp\left(\frac{1}{n}\sum_{i=2}^{n+1}\log\left(1+\varepsilon_i\log^{-a}(i)\right)\right) = \\ = 1 + \frac{1}{n}\sum_{i=2}^{n+1}\log\left(1+\varepsilon_i\log^{-a}(i)\right) + O\left(n^{-1}\right).$$

5. The Hurwitz identity

The last result of this paragraph will be preceded by quite extensive introduction, which goal is to proof some auxiliary identity (identity (12) below) obtained ones by Adolf Hurwitz. This identity was using by Hurwithz to prove A-G inequality. It can be also applied to the estimation of same asimptotic relations for interesting for us differences between arithmetical and geometrical means. Let $\mathfrak{F}_n$ be a family of all maps $f : \mathbb{R}^n \to \mathbb{R}$. Let $\sum : \mathfrak{F}_n \to \mathfrak{F}_n$ be an linear operator defined for all functions $f \in \mathfrak{F}_n$ by:

$$\sum(f) = \sum_{\sigma\in S_n} f(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(n)})$$

where S_n denotes the family of all permutations of the set $\{1, 2, \ldots, n\}$. Now, for all $x_i \in \mathbb{R}$, $i = 1, 2, \ldots, n$, $n \in \mathbb{N}$, let us assume:

$$\begin{aligned}
\phi(x_1, x_2, \ldots, x_n) &= \frac{1}{n}\left(x_1^n + x_2^n + \ldots + x_n^n\right) - x_1 x_2 \ldots x_n,\\
\phi_1(x_1, x_2, \ldots, x_n) &= \sum\left((x_1^{n-1} - x_2^{n-1})(x_1 - x_2)\right),\\
\phi_2(x_1, x_2, \ldots, x_n) &= \sum\left((x_1^{n-2} - x_2^{n-2})(x_1 - x_2)x_3\right),\\
\phi_3(x_1, x_2, \ldots, x_n) &= \sum\left((x_1^{n-3} - x_2^{n-3})(x_1 - x_2)x_3x_4\right),\\
&\vdots\\
\phi_{n-1}(x_1, x_2, \ldots, x_n) &= \sum\left((x_1 - x_2)(x_1 - x_2)x_3x_4\ldots x_n\right).
\end{aligned}$$

Theorem 17. *(A. Hurwitz) The following identity holds:*

$$2(n!)\phi(x_1, x_2, \dots, x_n) = \phi_1(x_1, x_2, \dots, x_n) + \\ + \phi_2(x_1, x_2, \dots, x_n) + \dots + \phi_{n-1}(x_1, x_2, \dots, x_n). \quad (12)$$

Proof. It is easy to check that

$$\sum (x_1^n) = (n-1)!(x_1^n + x_2^n + \dots + x_n^n) \quad (13)$$

and

$$\sum (x_1 x_2 \dots x_n) = n! x_1 x_2 \cdot \ldots \cdot x_n. \quad (14)$$

It is also true that if $p, q \in \mathfrak{F}_n$ and

$$p(x_1, x_2, \dots, x_n) = q(x_{\sigma(1)}, x_{\sigma(2)}, \dots, x_{\sigma(n)})$$

for some $\sigma \in S_n$ and for some $x_1, x_2, \dots, x_n \in \mathbb{R}$, then

$$\sum (p(x_1, x_2, \dots, x_n)) = \sum (q(x_1, x_2, \dots, x_n)).$$

Because of the facts mentioned above and linearity of operator $\sum$ we obtain the following identity:

$$\begin{aligned}
\phi_1(x_1, x_2, \dots, x_n) &= \sum \left(x_1^n + x_2^n - x_1^{n-1}x_2 - x_1 x_2^{n-1}\right) = \\
&= 2\sum \left(x_1^n\right) - 2\sum \left(x_1^{n-1}x_2\right), \\
\phi_2(x_1, x_2, \dots, x_n) &= 2\sum \left(x_1^{n-1}x_2\right) - 2\sum \left(x_1^{n-2}x_2x_3\right), \\
\phi_3(x_1, x_2, \dots, x_n) &= 2\sum \left(x_1^{n-2}x_2x_3\right) - 2\sum \left(x_1^{n-3}x_2x_3x_4\right), \\
&\dots \\
\phi_{n-1}(x_1, x_2, \dots, x_n) &= 2\sum \left(x_1^2 x_2 \dots x_{n-1}\right) - 2\sum \left(x_1 x_2 \dots x_n\right),
\end{aligned}$$

what after adding up both sides gives us

$$\phi_1(x_1, x_2, \dots, x_n) + \phi_2(x_1, x_2, \dots, x_n) + \dots + \phi_{n-1}(x_1, x_2, \dots, x_n) = \\ = 2\sum \left(x_1^n\right) - 2\sum \left(x_1 x_2 \dots x_n\right),$$

what finally, together with (13) and (14), leads to the expected relation (12). □

The identity (12) helps us to prove the following result.

Theorem 18. *Let* $\{\alpha_i\}_{i=1}^{\infty} \subset (0,\infty)$ *and* $\beta \in (0,\infty)$, $\alpha_{i+1} \geqslant \beta\alpha_i$, $i \in \mathbb{N}$. *Then the following inequality holds:*

$$\frac{A_n(\alpha_i)}{G_n(\alpha_i)} \geqslant \frac{1}{2n(n-1)} \sum_{k=1}^{n-1} (n-k)\sqrt[n]{\beta^k}. \tag{15}$$

Proof. By assumptions we have $\alpha_j \geqslant \beta^{j-i}\alpha_i$, $j \geqslant i$. Hence and from (12) results that

$$\begin{aligned}
2(n!)(A_n(\alpha_i) - G_n(\alpha_i)) > \varphi_{n-1}(\sqrt[n]{\alpha_1}, \sqrt[n]{\alpha_2}, \ldots, \sqrt[n]{\alpha_n}) = \\
= \sum \left(\sqrt[n]{\alpha_3}\,\sqrt[n]{\alpha_4}\cdot\ldots\cdot\sqrt[n]{\alpha_n}\cdot(\sqrt[n]{\alpha_1} - \sqrt[n]{\alpha_2})^2\right) = \\
= G_n(\alpha_i)\sum\left(\frac{(\sqrt[n]{\alpha_1} - \sqrt[n]{\alpha_2})^2}{\sqrt[n]{\alpha_1\alpha_2}}\right) = \\
= G_n(\alpha_i)\left(-2(n!) + \sum\left(\sqrt[n]{\frac{\alpha_1}{\alpha_2}} + \sqrt[n]{\frac{\alpha_2}{\alpha_1}}\right)\right) = \\
= 2G_n(\alpha_i)\left(-n! + \sum\left(\sqrt[n]{\frac{\alpha_1}{\alpha_2}}\right)\right) \geqslant \\
\geqslant 2G_n(\alpha_i)\left(-n! + (n-2)!\cdot\frac{1}{2}\sum_{k=1}^{n-1}(n-k)\sqrt[n]{\beta^k}\right)
\end{aligned}$$

which implies the inequality (15). □

Remark 19. If $\alpha_{i+1} \approx \beta\alpha_i$, $1 \leqslant i \leqslant n$, where $n \in \mathbb{N}$ is fixed then from the above proof the following relation follows:

$$\begin{aligned}
\varphi_{n-1}(\sqrt[n]{\alpha_1}, \sqrt[n]{\alpha_2}, \ldots, \sqrt[n]{\alpha_n}) \approx \\
\approx G_n(\alpha_i)(-2(n!) + (n-2)!\cdot\frac{1}{2}\sum_{k=1}^{n-1}(n-k)\left(\sqrt[n]{\beta^k} + \frac{1}{\sqrt[n]{\beta^k}}\right))
\end{aligned}$$

so the inequality (15) can be improved.

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Omówienie

W artykule przedstawiono przegląd tematyczny oraz wybrane wyniki dotyczące asymptotycznych zachowań ciągów średnich arytmetycznych i geometrycznych danych ciągów liczb dodatnich. W drugim rozdziale przedstawiono asymptotykę ciągów średnich arytmetycznych i geometrycznych ciągu kolejnych liczb pierwszych. Wyniki te częściowo uogólniono, stosując rzadko cytowane twierdzenie Kaleckiego. W rozdziale trzecim wyznaczono oszacowania granic dolnej i górnej ciągu

$\{a_n^{-1} G_n(a_i)\}$. Badana jest też asymptotyka pewnych ciągów szczególnych zwłaszcza $\{\sqrt[n]{n!}/n\}$, gdzie wykorzystano m.in. wzór Stirlinga i przede wszystkim ostatnie wyniki Morticiego i Parisa. W rozdziale czwartym badana jest asymptotyka ciągów średnich arytmetycznych i geometrycznych potęg logarytmów kolejnych liczb naturalnych oraz zaburzeń takich ciągów. Wreszcie w ostatnim rozdziale przypomniano tożsamość Hurwitza i zaproponowano jej wykorzystanie przy badaniu asymptotyki ilorazu $A_n(a_i)/G_n(a_i)$.