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AN OPERATIONAL CALCULUS MODEL FOR THE n^{TH} — ORDER FORWARD DIFFERENCE

ABSTRACT

In this paper, there has been constructed such a model of a non-classical Bittner operational calculus, in which the derivative is understood as a forward difference $\Delta_n\{x(k)\} := \{x(k+n) - x(k)\}$. Next, considering the operation $\Delta_{n,b}\{x(k)\} := \{x(k+n) - b\,x(k)\}$, the presented model has been generalized.

Key words:
operational calculus, derivative, integrals, limit conditions, forward difference.

THE NON-CLASSICAL BITTNER OPERATIONAL CALCULUS

The *Bittner operational calculus* [1–4]

$$CO(L^0, L^1, S, T_q, s_q, Q)^1 \tag{1}$$

is understood as a system, in which L^0 and L^1 are linear spaces (over a field Γ of scalars), such that $L^1 \subset L^0$. Moreover, a linear operation $S : L^1 \longrightarrow L^0$ (which is described as $S \in \mathscr{L}(L^1, L^0)$), called a *derivative*, is a surjection. Q is a set of indices q for the operations $T_q \in \mathscr{L}(L^0, L^1)$ and $s_q \in \mathscr{L}(L^1, L^1)$, called *integrals* and *limit conditions*, respectively. These operations have to fulfil the properties $S T_q f = f, f \in L^0$ and $s_q x = x - T_q S x, x \in L^1$. The kernel of S, i.e. $\operatorname{Ker} S$, is called a set of *constants* for the derivative S. The limit conditions $s_q, q \in Q$ are projections of L^1 on the subspace $\operatorname{Ker} S$.

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[1] The abbreviation *CO* is derived from the French *calcul opératoire* (operational calculus).

When speaking of a *representation* or a *model* of an operational calculus, we have in mind a system (1), in which all the objects are defined. A classic example of an operational calculus (1) is a *discrete* model with the derivative as a forward difference $\Delta\{x(k)\} := \{x(k+1) - x(k)\}$.

THE FORWARD DIFFERENCE MODEL

Let $\mathbb{N}_0$ and $\mathbb{C}$ mean sets of non-negative integers and complexes, respectively. Moreover, let $L^0 = L^1 := C(\mathbb{N}_0, \mathbb{C})$ be a linear space of complex sequences $x = \{x(k)\}_{k\in\mathbb{N}_0}$ with usual operations on sequences. In [1, 2, 4] Bittner considered a model with a derivative

$$S\,x \equiv \Delta x := \{x(k+1) - x(k)\}, \tag{2}$$

to which there was corresponding an integral

$$T_0 x := \begin{cases} 0 & \text{for} \quad k = 0 \\ \sum\limits_{i=0}^{k-1} x(i) & \text{for} \quad k > 0 \end{cases}, \quad k \in \mathbb{N}_0$$

and a limit condition

$$s_0 x := \{x(0)\},$$

where $x = \{x(k)\} \in L^0 = L^1$. Later there appeared, mentioned officially in [5], a model with the derivative (2), integrals

$$T_{k_0} x := \begin{cases} -\sum\limits_{i=k}^{k_0-1} x(i) & \text{for} \quad k < k_0 \\ 0 & \text{for} \quad k = k_0 \\ \sum\limits_{i=k_0}^{k-1} x(i) & \text{for} \quad k > k_0 \end{cases}, \quad k \in \mathbb{N}_0 \tag{3}$$

and limit conditions

$$s_{k_0} x := \{x(k_0)\}, \tag{4}$$

where $k_0 \equiv q \in Q := \mathbb{N}_0$[2] (see also [11]). This model was generalized in [7] by Mieloszyk. He proved that to the so-called *forward difference on the basis* $b = \{b(k)\}$, i.e.

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[2] Due to the definition of T_{k_0}, we assume that $\sum_{i=k_0}^{k_0-1} x(i) := 0$.

$$S_b x := \{x(k+1) - b(k)x(k)\},$$

where $b(k) \neq 0$ for each $k \in \mathbb{N}_0$ and $\{b(k)\}\{x(k)\} := \{b(k)\,x(k)\}$ means the usual multiplication of sequences b, x in the algebra L^0, there correspond the integrals

$$T_{b,k_0}x = \{e(k)\}T_{k_0}\left\{\frac{x(k)}{e(k+1)}\right\}$$

and limit conditions

$$s_{b,k_0}x = \left\{\frac{e(k)}{e(k_0)}\right\} s_{k_0}\{x(k)\},$$

where $e(k) := \prod_{i=0}^{k-1} b(i), e(0) := 1$.

Having in mind further considerations, let us notice that the integrals (3) can be presented in the concise form of

$$T_{k_0}x = \left\{\sum_{i=0}^{k-1} x(i) - \sum_{i=0}^{k_0-1} x(i)\right\}. \tag{5}$$

A HIGHER ORDER FORWARD DIFFERENCE MODEL

A generalization of the operation $\Delta \equiv \Delta_1$ is the n^{th} — *order forward difference*

$$\Delta_n\{x(k)\} := \{x(k+n) - x(k)\}, \tag{6}$$

where n is a given natural number.

We will determine integrals T_{k_0} and limit conditions s_{k_0} corresponding to (6) understood as the derivative S. Firstly, let us notice that an arbitrary constant c for (6) is an *n-periodic sequence*, i.e. it satisfies the condition $c(k+n) = c(k)$ for each $k \in \mathbb{N}_0$. What is more, for any sequence $c \in \operatorname{Ker}\Delta_n$ there exist numbers $a_0, a_1, \ldots, a_{n-1} \in \mathbb{C}$ such that

$$c = \{a_0\varepsilon_0^k + a_1\varepsilon_1^k + \cdots + a_{n-1}\varepsilon_{n-1}^k\}, \tag{7}$$

where

$$\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_{n-1} \tag{8}$$

are n^{th} roots of unity, i.e.

$$\varepsilon_j = \cos\frac{2j\pi}{n} + \mathrm{i}\,\sin\frac{2j\pi}{n}, \quad j \in \overline{0, n-1}^{3},$$

while 'i' is the imaginary unit.

In what follows, we shall use the below sequence (8) properties:

$$\varepsilon_j^{k+n} = \varepsilon_j^k, \quad j \in \overline{0, n-1}, k \in \mathbb{N}_0,$$

$$\varepsilon_0^m + \varepsilon_1^m + \ldots + \varepsilon_{n-1}^m = 0, \quad m \neq \ell n, \ell, m \in \mathbb{Z}^4, n \in \mathbb{N} \setminus \{1\}.$$

We shall prove the following:

Theorem. *The system* (1), where $x = \{x(k)\} \in L^0 = L^1 := C(\mathbb{N}_0, \mathbb{C}), k_0 \equiv q \in Q := \mathbb{N}_0$ *and*

$$S\,x := \{x(k+n) - x(k)\}, \tag{9}$$

$$T_{k_0}x := \left\{\frac{1}{n}\sum_{j=0}^{n-1}\Big[\sum_{i=0}^{k-1}\varepsilon_j^{k-i}\,x(i) - \sum_{i=0}^{k_0-1}\varepsilon_j^{k-i}\,x(i)\,\Big]\right\}, \tag{10}$$

$$s_{k_0}x := \left\{\frac{1}{n}\sum_{j=0}^{n-1}\sum_{i=k_0}^{k_0+n-1}\varepsilon_j^{k-i}\,x(i)\right\} \tag{11}$$

forms a discrete Bittner operational calculus model[5].

Proof. It is obvious that operations (9)–(11) are linear. Let $\{y(k)\} := T_{k_0}\{x(k)\}$. Then

$$S\{y(k)\} = \{y(k+n) - y(k)\}$$
$$= \left\{\frac{1}{n}\sum_{j=0}^{n-1}\Big[\sum_{i=0}^{k+n-1}\varepsilon_j^{k+n-i}\,x(i) - \sum_{i=0}^{k_0-1}\varepsilon_j^{k+n-i}\,x(i)\,\Big]\right.$$
$$\left. -\frac{1}{n}\sum_{j=0}^{n-1}\Big[\sum_{i=0}^{k-1}\varepsilon_j^{k-i}\,x(i) - \sum_{i=0}^{k_0-1}\varepsilon_j^{k-i}\,x(i)\,\Big]\right\} = \left\{\frac{1}{n}\sum_{j=0}^{n-1}\sum_{i=k}^{k+n-1}\varepsilon_j^{k-i}\,x(i)\right\}.$$

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3 $\overline{0, n-1} := \{0, 1, \ldots, n-1\}$.

4 $\mathbb{Z}$ denotes the set of integers.

5 We assume that $\sum_{i=0}^{-1} x(i) := 0$.

It is not difficult to notice that for $n = 1$ we have $S\{y(k)\} = \{x(k)\}$, while for $n > 1$ we can write

$$S\{y(k)\} = \{x(k)\} + \frac{1}{n}\left\{\sum_{i=k+1}^{k+n-1}[\varepsilon_0^{k-i} + \varepsilon_1^{k-i} + \cdots + \varepsilon_{n-1}^{k-i}]x(i)\right\} = \{x(k)\}. \tag{12}$$

Finally, it can be stated that the property $S\,T_{k_0}\{x(k)\} = \{x(k)\}$ is satisfied.

Let $\{f(k)\} := S\{x(k)\} = \{x(k+n) - x(k)\}$. Then

$$T_{k_0}S\{x(k)\} = T_{k_0}\{f(k)\} = \left\{\frac{1}{n}\sum_{j=0}^{n-1}[\sum_{i=0}^{k-1}\varepsilon_j^{k-i}f(i) - \sum_{i=0}^{k_0-1}\varepsilon_j^{k-i}f(i)]\right\}$$

$$= \left\{\frac{1}{n}\sum_{j=0}^{n-1}[\sum_{i=0}^{k-1}\varepsilon_j^{k-i}[x(i+n) - x(i)] - \sum_{i=0}^{k_0-1}\varepsilon_j^{k-i}[x(i+n) - x(i)]]\right\}$$

$$= \left\{\frac{1}{n}\sum_{j=0}^{n-1}[\sum_{i=n}^{k+n-1}\varepsilon_j^{k-i}x(i) - \sum_{i=0}^{k-1}\varepsilon_j^{k-i}x(i) - \sum_{i=n}^{k_0+n-1}\varepsilon_j^{k-i}x(i) + \sum_{i=0}^{k_0-1}\varepsilon_j^{k-i}x(i)]\right\}$$

$$= \left\{\frac{1}{n}\sum_{j=0}^{n-1}[\sum_{i=n}^{k+n-1}\varepsilon_j^{k-i}x(i) + \sum_{i=0}^{n-1}\varepsilon_j^{k-i}x(i) - \sum_{i=0}^{k-1}\varepsilon_j^{k-i}x(i)\right.$$

$$\left. - (\sum_{i=n}^{k_0+n-1}\varepsilon_j^{k-i}x(i) + \sum_{i=0}^{n-1}\varepsilon_j^{k-i}x(i) - \sum_{i=0}^{k_0-1}\varepsilon_j^{k-i}x(i))]\right\}$$

$$= \left\{\frac{1}{n}\sum_{j=0}^{n-1}\sum_{i=k}^{k+n-1}\varepsilon_j^{k-i}x(i)\right\} - \left\{\frac{1}{n}\sum_{j=0}^{n-1}\sum_{i=k_0}^{k_0+n-1}\varepsilon_j^{k-i}x(i)\right\}.$$

By analogy to (12), we eventually get

$$T_{k_0}S\{x(k)\} = \{x(k)\} - \left\{\frac{1}{n}\sum_{j=0}^{n-1}\sum_{i=k_0}^{k_0+n-1}\varepsilon_j^{k-i}x(i)\right\}.$$

So the property $T_{k_0}S\{x(k)\} = \{x(k)\} - s_{k_0}\{x(k)\}$ is also fulfilled. □

Let us observe that (2), (4), (5) constitute a particular case of the above model for $n = 1$.

Example. The limit condition (11) allows to present an arbitrary n-periodic sequence $c = \{c(k)\}$ with a recurring cycle $(c_0, c_1, \ldots, c_{n-1})$, i.e.

$$c = \{(c_0, c_1, \ldots, c_{n-1})\} := \{c_0, c_1, \ldots, c_{n-1}, c_0, c_1, \ldots, c_{n-1}, \ldots\}$$

in the form of (7). For, if $c \in \operatorname{Ker} \Delta_n$, we have $s_{k_0} c = c$. Then for $k_0 = 0$ we get

$$c = \left\{ \frac{1}{n} \sum_{j=0}^{n-1} \sum_{i=0}^{n-1} \varepsilon_j^{k-i} c_i \right\}. \tag{13}$$

A lot of interesting examples of periodic sequences are included in *The On--Line Encyclopedia of Integer Sequences* OEIS®[6]. Some of them are related to the *Fibonacci sequence* $\{F(k)\}$, whose terms meet the conditions

$$F(k+1) = F(k) + F(k-1), \quad k \in \mathbb{N}$$

and $F(0) = 0, F(1) = 1$.

In 1960, Wall proved [9] that for each natural number $m > 1$, the sequence

$$\{F_m(k)\} := \{F(k) \pmod{m}\} \tag{14}$$

is $p(m)$-periodic[7].

Thus, if in (13) we take $n := p(m)$ and $c_i := F_m(i)$, then

$$F_m(k) = \frac{1}{p(m)} \sum_{j=0}^{p(m)-1} \sum_{i=1}^{p(m)-1} \varepsilon_j^{k-i} F_m(i), \quad k \in \mathbb{N}_0. \tag{15}$$

The below table contains sequences (14) chosen from OEIS® as well as trigonometric forms of their general terms obtained on the basis of (15) by using the *Mathematica*® program.

[6] https://oeis.org/.
[7] The number $p(m)$ is called the *Pisano period* of the sequence (14) [10]. The Pisano periods for $m \leqslant 10 \cdot 10^7$ can be calculated directly using the Marc Renault [8] web browser applet available at http://webspace.ship.edu/msrenault/fibonacci/fibfactory.htm.

OEIS®	m	$p(m)$	Cycle	$F_m(k)$
A011655	2	3	(0, 1, 1)	$\frac{4}{3}\sin^2\frac{k\pi}{3}$
A082115	3	8	(0, 1, 1, 2, 0, 2, 2, 1)	$\frac{1}{8}\Big(9-2\sqrt{2}\cos\frac{k\pi}{4}-6\cos\frac{k\pi}{2}$ $+2\sqrt{2}\cos\frac{3k\pi}{4}-3\cos k\pi$ $-2\sin\frac{k\pi}{4}+2\sin\frac{3k\pi}{4}\Big)$
A079343	4	6	(0, 1, 1, 2, 3, 1)	$\frac{2}{3}\sin\frac{k\pi}{6}\Big(\sqrt{3}\cos\frac{k\pi}{2}$ $+4\sin\frac{k\pi}{6}+\sin\frac{k\pi}{2}\Big)$
A079344	8	12	(0, 1, 1, 2, 3, 5, 0, 5, 5, 2, 7, 1)	$\frac{2}{3}\sin\frac{k\pi}{6}\Big(2\sqrt{3}\cos\frac{k\pi}{3}+\sqrt{3}\cos\frac{k\pi}{2}+2\sqrt{3}\cos\frac{2k\pi}{3}$ $+8\sin\frac{k\pi}{6}+(5-4\sqrt{3}\cos\frac{k\pi}{6})\sin\frac{k\pi}{2}\Big)$

SOME GENERALIZATION

The operation

$$S_b\{x(k)\} := \{x(k+n) - bx(k)\}, \tag{16}$$

where $\{x(k)\} \in L^0 = L^1 := C(\mathbb{N}_0, \mathbb{C}), b \in \mathbb{C} \setminus \{0\}$, is a generalization of the derivative (9).

In order to construct an operational calculus model related to the derivative (16), we will use the idea of solving the equation $x(k+1) - b(k)\,x(k) = f(k)$ described in [6] as well as the following auxiliary theorems:

Lemma 1 (Th. 3 [4]). *An abstract differential equation*

$$Sx = f, \quad f \in L^0, x \in L^1$$

with the limit condition

$$s_q x = x_{0,q}, \quad x_{0,q} \in \operatorname{Ker} S$$

has exactly one solution

$$x = x_{0,q} + T_q f. \tag{17}$$

Lemma 2 (Th. 4 [4]). *With a given derivative $S \in \mathscr{L}(L^1, L^0)$, the projection $s_q \in \mathscr{L}(L^1, \operatorname{Ker} S)$ determines the integral $T_q \in \mathscr{L}(L^0, L^1)$ from the condition*

$$x = T_q f \quad \textit{if and only if} \quad S x = f, s_q x = 0.$$

Moreover, s_q is a limit condition corresponding to the integral T_q.

Let us notice that one of the elements of the space $\operatorname{Ker} S_b$ is the sequence

$$e(k) := b^{\frac{k}{n}}, \quad k \in \mathbb{N}_0.$$

So

$$e(k+n) = be(k), \quad k \in \mathbb{N}_0.$$

Let us consider the difference equation

$$S_b\{x(k)\} = \{f(k)\},$$

i.e.

$$x(k+n) - bx(k) = f(k), \quad k \in \mathbb{N}_0. \tag{18}$$

Hence we get

$$\frac{x(k+n)}{e(k+n)} - \frac{x(k)}{e(k)} = \frac{f(k)}{e(k+n)}, \quad k \in \mathbb{N}_0,$$

that is

$$y(k+n) - y(k) = g(k), \quad k \in \mathbb{N}_0, \tag{19}$$

where

$$y(k) := \frac{x(k)}{e(k)}, \quad g(k) := \frac{f(k)}{e(k+n)}, \quad k \in \mathbb{N}_0. \tag{20}$$

The equation (19) can be presented in the form of

$$S\{y(k)\} = \{g(k)\}, \tag{21}$$

where $S \equiv \Delta_n$ is the operation (9).

From Lemma 1 it follows that the sequence

$$\{y(k)\} = s_{k_0}\{y(k)\} + T_{k_0}\{g(k)\},$$

where T_{k_0} and s_{k_0} are operations (10) and (11), is the solution of the equation (21).

From (20) we get $x(k) = e(k)\, y(k), k \in \mathbb{N}_0$. Finally,

$$\{x(k)\} = \{e(k)\}s_{k_0}\left\{\frac{x(k)}{e(k)}\right\} + \{e(k)\}T_{k_0}\left\{\frac{f(k)}{e(k+n)}\right\} \tag{22}$$

constitutes the solution of the equation (18).

If we take

$$\{\widetilde{c}(k)\} := s_{k_0}\left\{\frac{x(k)}{e(k)}\right\},$$

then the sequence $\{\widetilde{c}(k)\} \in \operatorname{Ker} S$ is n-periodic, i.e.

$$\widetilde{c}(k+n) = \widetilde{c}(k), \quad k \in \mathbb{N}_0.$$

Let

$$s_{b,k_0}\{x(k)\} := \{e(k)\}s_{k_0}\left\{\frac{x(k)}{e(k)}\right\}, \quad k_0 \in Q := \mathbb{N}_0, \{x(k)\} \in L^1. \tag{23}$$

Thus, for each $k \in \mathbb{N}_0$ we obtain

$$\begin{aligned} S_b s_{b,k_0} x(k) &= e(k+n)\widetilde{c}(k+n) - b\, e(k)\widetilde{c}(k) \\ &= e(k+n)(\widetilde{c}(k+n) - \widetilde{c}(k)) = e(k+n)\cdot 0 = 0, \end{aligned}$$

which means that $s_{b,k_0} \in \mathscr{L}(L^1, \operatorname{Ker} S_b)$. Moreover, for each $k \in \mathbb{N}_0$ holds the below

$$\begin{aligned} s^2_{b,k_0} x(k) &= s_{b,k_0}[e(k)\widetilde{c}(k)] = e(k)s_{k_0}\left[\frac{e(k)\widetilde{c}(k)}{e(k)}\right] \\ &= e(k)s_{k_0}\widetilde{c}(k) = e(k)\widetilde{c}(k) = s_{b,k_0}x(k), \end{aligned}$$

since $s_{k_0}\{\widetilde{c}(k)\} = \{\widetilde{c}(k)\}$. Eventually, s_{b,k_0} is a projection of L^1 onto $\operatorname{Ker} S_b$ for each $k_0 \in \mathbb{N}_0$. From Lemma 2 it follows that the projection s_{b,k_0} determines the *integral* T_{b,k_0} from the formula (22). Namely,

$$T_{b,k_0}\{f(k)\} := \{e(k)\}T_{k_0}\left\{\frac{f(k)}{e(k+n)}\right\}, \quad k_0 \in Q, \{f(k)\} \in L^0. \tag{24}$$

What is more, s_{b,k_0} is the *limit condition* corresponding to the integral (24). Hence we arrive at the

Corollary. *The system* (16), (23), (24) *constitutes a discrete model of the Bittner operational calculus*

$$CO(C(\mathbb{N}_0,\mathbb{C}),C(\mathbb{N}_0,\mathbb{C}),S_b,T_{b,k_0},s_{b,k_0},\mathbb{N}_0).$$

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MODEL RACHUNKU OPERATORÓW DLA RÓŻNICY PROGRESYWNEJ RZĘDU *n*

STRESZCZENIE

W artykule skonstruowano model nieklasycznego rachunku operatorów Bittnera, w którym pochodna rozumiana jest jako różnica progresywna $\Delta_n\{x(k)\} := \{x(k+n) - x(k)\}$. Następnie dokonano uogólnienia opracowanego modelu, rozważając operację $\Delta_{n,b}\{x(k)\} := \{x(k+n) - b\,x(k)\}$.

Słowa kluczowe:
rachunek operatorów, pochodna, pierwotne, warunki graniczne, różnica progresywna.