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FILTRATION OF DIGITAL SIGNALS USING WAVELETS

The article constitutes a set of research referring to the removal of the noise of Gaussian distribution of signal using wavelet analysis. The thesis has been launched of presentation of existing analysis algorithms, starting of discrete transform related to Mallat's algorithm, and ending with the selected methods on inverse wavelet transform. The task was carried out by using the simulation scripts of package Wavelet Toolbox in MATLAB. The studies of using on-line filtering algorithms has been undergone. The delay value, which was implemented by the various filtration methods, was determined through simulations. The inclusion of the comparison of results allows to evaluate the quality of filtration.

KEYWORDS: wavelet transform, on-line filtering, Gaussion noise, digital signal

1. INTRODUCTION TO WAVELET TRANSFORM

This paper presents on-line filtration utilising two algorithms of discrete wavelet transform. The filtration will be aimed at removing the noise signal of a normal distribution. Study subjected algorithms are the algorithms of discrete and stationary wavelet transform. A key factor assessment of filtration algorithm is the quality of signal reflection.

1.1. Continuous Wavelet Transform

The Continuous Wavelet Transform (CWT) is frequently compared with Fourier Transform, but the main difference is a kernel transformation. The kernel of wavelet transform is a special function (wavelet), whereas Fourier transform is using sinusoidal functions as kernel. As a result of continuous wavelet transform there are coefficients determining the similarity between the signal and the selected wavelet. The continuous signals transform is an integral transformation represented by [1]:

$$Wf\left(a,b\right)=\frac{1}{\sqrt{a}}\int_{-\infty}^{\infty} f\left(t\right)\Psi\left(\frac{t-b}{a}\right)dt, \tag{1}$$

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where: a – scale parameter, $a \in R^{+}$, b – shift parameter, $b \in R$, *f(t)* – analyzed signal, depending on time t, *Wf(a)* – wavelet coefficient, depending on parameters a and b, $\frac{1}{\sqrt{a}}\Psi\left(\frac{t-b}{a}\right)$ – transformation kernel.

According to formula (1) wavelet analysis consists of finding values of amplitude products, in terms of time - frequency coefficients a and b. The rank of compliance of a particular function with waveform $f(t)$ is presented by the values of coefficients within a specified time period. The kernel transformation is being generated based on basic wavelet through dilatation coefficient *b* and scale coefficient *a*. The value of b is equal to the basic wavelets shift in time. Otherwise the big values of a ($a >> 1$) refer to „wide" basic functions, in view of their isolation of long-term characteristics of signal dynamics, especially behavior in the steady state. The small values are responsible for „narrow" basic functions, which enable the extraction of short-term behavior of the signal. The factor $\frac{1}{\sqrt{a}}$ ensures that the energy of wavelet does not change with a change of scale [1].

1.2. Discrete Wavelet Transform

Discrete Wavelet Transform (DWT), utilizing Mallat's algorithm [2], is a decomposition of a signal into parts: approximation and detail. The detail and approximation are results of filtering with pair of filters (mirror quadrature filters). The filters, lowpass and highpass (in that order) are calculated on the basis of scaling and wavelet functions. Due to performing the filtering with lowpass filter, the approximation is obtained (elements of a signal of low frequency), instead of highpass filtering returning the detail (elements of a signal of high frequency). As a consequence of these operations on the first decomposition's level two signals with the same number of samples (N), as input signal, are obtained. It would generate a large amount of data for a large number of subdivisions of the signal. However, the signal obtained from the low-frequency filter has the highest frequency, equal to half of the frequency of the original signal. Using the Nyquist sampling theorem, it is known that for signal reproduction it is required to have a sampling frequency equal to half of the maximum frequency. It allows to remove every other sample and consequently reduce the number of generated data (decimation). Another simulation through a set of filters is equivalent to a reduction of a temporal resolution ($\frac{N}{2}$). However, it increases the frequency resolution (doubling the scale) [2, 3, 4,]. Decomposition algorithm is shown in Figure (Fig. 1).

The signal $x(t)$ can be presented in Formula [5]:

$$x(t)=\sum_{n} c_{m-k,n}\Phi\left(2^{m-k}t-n\right)+\sum_{i=0}^{k}\sum_{n} d_{m-i,n}\Psi\left(2^{m-i}t-n\right), \quad (2)$$

where: $c_{m-k,n}$ – approximation coefficients, $d_{m-i,n}$ – detail coefficients, k – level of decomposition, n – number of sample, Ψ – wavelet function, Φ – scale function.

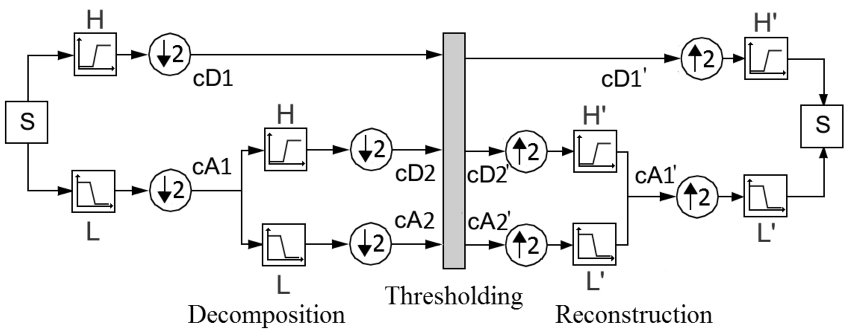

Fig. 1. Diagram of filtering using DWT and IDWT [6]

1.3. Stationary Wavelet Transform

Stationary Wavelet Transform (SWT) is a DWT modification. SWT is inherit redundant process, because each signal contains the same amount of data as the input data (decimation of DWT removal). It means, that on the N - level of decomposition there are $2N$ data to obtain. The subsequent modification is increasing the rank of filters on each level of decomposition (upsampling $N2^{j-1}$, j – level of decomposition) [4]. One of the methods of increasing rank of filter is to insert zeros between the filter coefficients. An exemplary block diagram of decomposition, based on the SWT algorithm is presented in the Fig. 2.

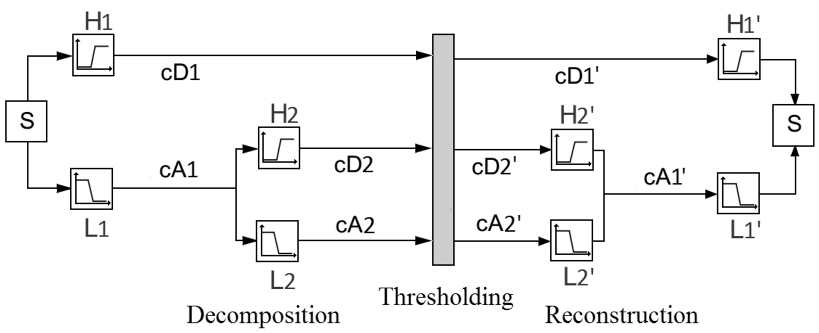

Fig. 2. Diagram of filtering using SWT and ISWT [6]

1.4. Inverse Discrete Wavelet Transform

Inverse Discrete Wavelet Transform (IDWT) enables signal transfer from the time-scale- ratio value system to the time-value system. The wavelet transform is a lossless conversion. All of the energy is saved during the transition of the signal from the time domain into a wavelet coefficients and vice versa. Equality defining discrete inverse transform:

$$f(t)=\sum_{m,n}(f,\psi_{mn})\psi_{mn}=\sum_{m}\sum_{n}d_m[n]\psi_{mn}. \tag{3}$$

In the foregoing formula, wavelet coefficients $d_m[n]=(f,\psi_{mn})$ determine common features of the signal f and the wavelet ψ_{mn}. The greater the value of wavelet coefficient is, the greater the correlation (the degree of compliance) between the waveform of the signal wavelet, and wavelet, with length defined by m and position defined by n and m is [1]. The increase of the number of signal samples takes place (upsampling) during the reconstruction process. It ends up with entering a zero as the second sample signal [5].

Inverse Stationary Wavelet Transform (ISWT), similar to IDWT enables the transition of the signal from the time-scale-ratio value system to the time-value system. Filters on each level synthesis reduce its rank in the opposite way than it happened for SWT.

2. THRESHOLD ELIMINATION AND EVALUATION OF QUALITY

2.1. Threshold elimination methods

Signal filtering takes place in the decomposed signal details. It involves the purging all signal items which do not meet a given condition. In this case, the condition is determined with a function from a selected threshold. The threshold can be calculated on the basis of the signal or adapted individually.

Typically, in the threshold elimination either hard or soft threshold is used. Above-mentioned methods are often used due to the simplicity of the calculations. Unfortunately, for specific purposes these thresholds may not be sufficient, because of their unfavorable properties. Hard threshold elimination is given by expression:

$$\delta_\lambda(d)=\begin{cases}d, & dla\ \ |d|>\lambda\\ 0, & dla\ \ |d|\le\lambda\end{cases}. \tag{4}$$

Soft threshold elimination, which solves the problem with discontinuity of functions is determined by Formula:

$$\delta_\lambda(d) = \begin{cases} 0, & dla \quad |d| \le \lambda \\ d-\lambda, & dla \quad d > \lambda \\ d+\lambda, & dla \quad d < -\lambda \end{cases}. \tag{5}$$

As a consequence of signal transition, the original function waveform disturbance occurs, an example of this is undesirable color change in image compression [7].

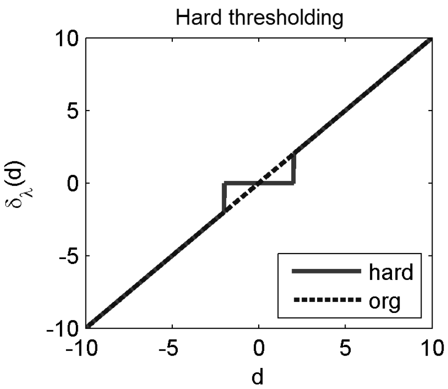

Fig. 3. Hard threshold elimination, threshold value: λ = 2

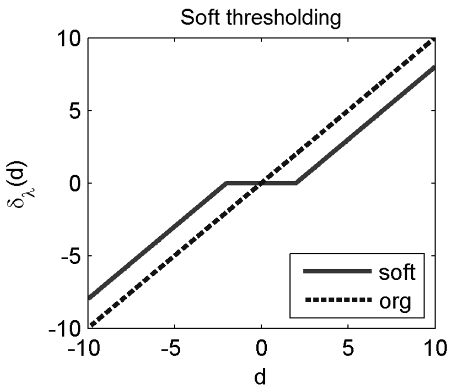

Fig. 4. Soft threshold elimination, threshold value: λ = 2

2.2. Threshold designation methods

There are three types of threshold: the global threshold, local threshold or local multi-distributing threshold. The global threshold assumes the value λ on every approximation level once the parameters when input signal are known. Using such threshold is quite convenient in practice, but less effective. The local threshold assumes a different value for each level of decomposition. The size of threshold is calculated on the basis of coefficients of details. The local multi-distributing threshold assumes different values for fixed detail coefficients. In other words for the details in the range of $< d_1; d_2 >$ a different threshold is used than for the range $< d_3; d_4 >$ [7].

One of the most common way of determining threshold is to use a universal threshold, where threshold value is strictly dependent on normalized standard deviation of details on each decomposition level [7].

$$\lambda = \sqrt{2\log n}\,\frac{\sigma}{\sqrt{n}}, \tag{6}$$

$$\sigma^2 = E\left\{\left[d - E(d)\right]^2\right\} = \sum_{i=1}^{n}(d_i - \bar{d})^2, \tag{7}$$

where: n – number of samples, d – detail value, σ – detail standard deviation.

Across all simulations performed, the local threshold was used, determined by detail coefficients of subsequent decomposition level.

2.3. Evaluation of filtering quality

For the assessment of the filtration it is necessary to introduce signal quality ratios. These are the scalar measurements that allow the easier interpretation and comparative analysis. Let the original digital signal of length N be the following function $f(n),\ \ 0 \le n < N$ if the reconstructed signal was determined by $\tilde{f}(n)$. The most useful scalar measurement of quality should include measurements such as [8]:

– Average Difference:

$$AD = \frac{1}{N}\sum_n \left| f(n) - \tilde{f}(n) \right|, \tag{8}$$

– Signal Fidelity:

$$SF = \left(1 - \frac{\sum_n \left[f(n) - \tilde{f}(n) \right]^2}{\sum_n \left[f(n) \right]^2} \right) \cdot 100\%, \tag{9}$$

– Maximum Difference:

$$MD = \max_n \{ | f(n) - \tilde{f}(n) | \}, \tag{10}$$

– Mean Square Error:

$$MSE = \frac{1}{N}\sum_n [f(n) - \tilde{f}(n)]^2. \tag{11}$$

3. FILTRATION USING WAVELET ANALYSIS

3.1 SWT and DWT methods. On-line filtering

The on-line filtering uses the algorithms of stationary or discrete wavelet transform. Analyzing the effect of the first method (DWT), the signal transition of the filters of decomposition can be determined. Subsequently, the reduction of sampling (*decimation*) is performed. Depending on the amount of decompositions and the length of the filter in the process, there is an adequate amount of delay. Based on these signals, the values for the selected threshold function are determined. Then the number of samples is increased, eventually the signal is subjected to the reconstruction carried out by using a synthesis filter.

Unlike the DWT, stationary wavelet transform (SWT) method is characterized by a higher pair rank of filters at every level of decomposition and reconstruction. The essence of this difference comes down to the output signal delay. For DWT, the signal offset value relative to the amount of decomposition and the length of the filter (N) is given by [9]:

$$\varphi_D = (N+1)\cdot(2^{dekom}-1). \tag{12}$$

For SWT on-line filtering method has also been designated dependence associated with the signal transition. Despite the fact that the length of the filter is doubled at each level, it has no impact on the overall delay of the output signal, because only the length of the filter on the first level (N_1) is taken into account [9].

$$\varphi_D = (N_1-1)\cdot(2^{dekom}-1). \tag{13}$$

Examples of filtration using DWT and SWT are presented in: Figs. 5 – 8. Using wavelet *symlets* (*sym9*), which stands out much stronger symmetry (in terms of axis of symmetry), than remaining wavelets, filtering using both methods was made. The results are summarized in Table 1.

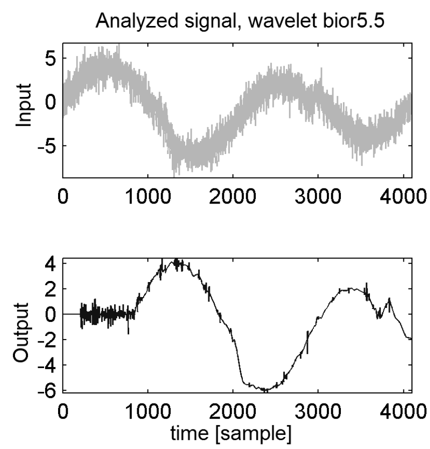

Fig. 5. Input and output from the model for the DWT algorithm

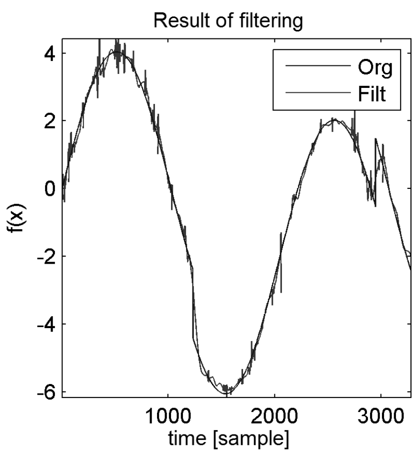

Fig. 6. The imposition of signals after 'heavy sine' filtering with the noise of a normal distribution and standard deviation = 3

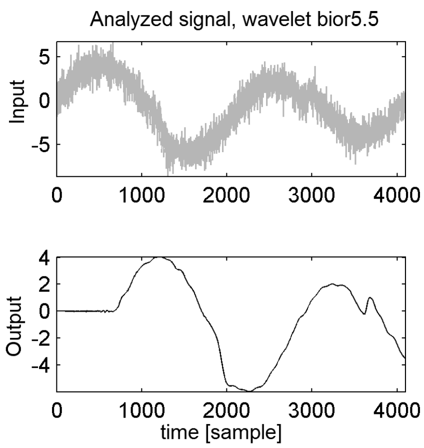

Fig. 7. Input and output from the model for the SWT algorithm

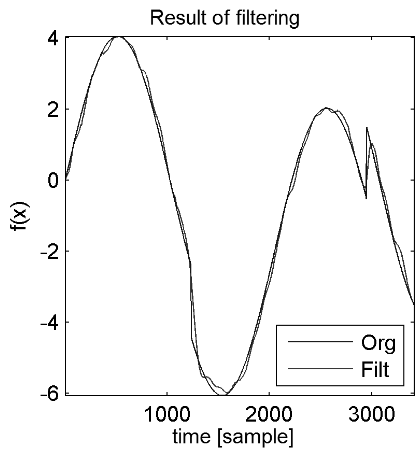

Fig. 8. The imposition of signals after 'heavy sine' filtering with the noise of a normal distribution and standard deviation = 3

Table 1. Result table for the signal 'heavy sine', with wavelet *'sym 9'* and a soft threshold function

Algorithm	Quality ratio	Number of decomposition					
		1	2	3	4	5	6
DWT	SF [%]	77,38	83,70	88,21	90,95	90,90	94,45
	AD [-]	0,562	0,407	0,295	0,222	0,202	0,130
	MD [-]	2,746	2,277	1,661	1,868	2,161	1,212
	MSE [-]	0,500	0,262	0,139	0,083	0,081	0,033
	φ_D [sample]	19	59	133	281	577	1199
SWT	SF [%]	78,22	84,58	89,03	92,26	91,54	**95,01**
	AD [-]	0,544	0,387	0,275	0,192	0,186	**0,103**
	MD [-]	2,629	1,855	1,203	1,593	1,961	**1,140**
	MSE [-]	0,463	0,234	0,120	0,061	0,071	**0,026**
	φ_D [sample]	16	52	119	252	513	**1073**

4. SUMMARY

In this paper two methods for filtration of signals using wavelet transform were presented. The elaborate analysis of the ratios' quality leads to the conclusion that better results were obtained for the SWT algorithm. Is not only the fidelity better, but also the delay across the filtration process is reduced. The most important is to choose a parameter with an appropriate signal decomposition depth, because approximation may lead to less accurate reflection. In the on-line filtering, the delay value is dependent on the depth of analysis and the filter length of the first decomposition. The basic criterion for this assessment was the quality of the signal reflection and the delay for on-line filtering methods.

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